CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T Th 9:00am
Lab – F 10:30am

Lecture Meeting Location: SP 309
Lab Meeting Location: SP 309

Business

• Reading: Please read Ch.2.1 and 2.2.2 in your textbook

• HW1 extended—due Sept. 17
  – Feel free to turn it in today, if you’d like

• HW2-Lookahead out today
  – Full HW2 out soon, due Sept. 24

• Lab0 due by the end of the day, Thursday, Sept. 17
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach
Empty Set

• The empty set, written as $\emptyset$, is the (unique) set containing no elements (Do you see what makes it unique?)
• The empty set can be described many ways, sometimes without it being obvious that it’s the empty set. E.g.,
  – $S = \{n \text{ is prime} | 24 \leq n \leq 28\}$
  – $G = \{\text{Grammy awards won by The Beach Boys, Led Zeppelin, The Who, The Doors, Queen, Guns N’ Roses, or Bob Marley during their careers (before 2014)}\}$ ignoring Lifetime Achievement Grammy awards, etc.
• Exercise: Consider a set $S$. If $\emptyset \subseteq S$, what do we know about $S$?
  – Prove your answer; it should contain a proof of an important fact about the empty set!

Disjoint Sets

• Definition: Sets $A, B$ are disjoint if there is no element $x$ s.t. ($x \in A \land x \in B$).
  – That is, sets $A$ and $B$ have no element in common
• This idea can be useful when talking about more than 2 sets where no two have an element in common—they’re all disjoint from each other
• Definition: A collection of sets $A_1, A_2, \ldots, A_n$ is pairwise disjoint iff for any $i, j \leq n$, $A_i$ and $A_j$ are disjoint
Set Difference

- Another set operation: Difference
  - Intuitively, the difference between two things is what’s in the first that’s not in the second
  - For sets A, B, the difference A - B (the book writes it as A \ B) is the set of elements that are in A but not in B. More formally...

- Definition: The difference A - B of sets A and B is defined by:
  \[ x \in A - B \text{ iff } x \in A \text{ and } x \notin B \]
  - Also written as A - B = \{ x | x \in A \text{ and } x \notin B \}

- Examples:
  - What’s \{1,2,3,4\} - \{2,4,6,8\}? 
  - What’s \{x \mid x \text{ is odd between 0 and 10}\} - \{x \mid x \text{ is prime between 0 and 10}\}

Enumerating Subsets

- If you’re given a set, especially a finite set, you might consider all the subsets of that set
  - If S = \{Phil Collins, Peter Gabriel\}, what are all its subsets? How many are there?
  - If S = \{1,2,3\}, what are all its subsets? How many are there?
  - If S = \{2,4,6,8\}, what are all its subsets? How many are there?

- What do you notice about the relationship between the number of elements in a set and the number of subsets it has?
Power Sets

- Given a set $S$, the set of subsets of $S$ is called the power set of $S$, sometimes written $\mathcal{P}(S)$
  - $\mathcal{P}(S) = \{ A \mid A \subseteq S \}$

- Questions:
  - What’s the power set of $\{a,b\}$?
  - Let $S$ be $\{x \mid x$ is between 0 and 10 and $x$ is prime\}.
    What is the size of the power set of $S$?
  - What is the size of $\mathcal{P}(\emptyset)$? What is $\mathcal{P}(\emptyset)$?

Complement of a Set

- Another set operation: Complement
  - For set $A$, the complement $\overline{A}$ is the set of elements not in $A$. More formally…
  - Definition: For set $A$, its complement $\overline{A}$ (also written as $A$ with a line over it) is defined by:
    - $x \in \overline{A}$ iff $x \notin A$
    - Also written as $\overline{A} = \{ x \mid x \notin A \}$
  - When discussing set complement, it is (explicitly or implicitly) in the context of a universe $U$ of all elements to be considered
    - So, $\overline{A} = U - A$
  - Examples:
    - What’s $\overline{\{1,2,3,4\}}$? [Assume that the complement here is with respect to the natural numbers]
Generalized Union / Intersection

- Consider the $\sum$ summation notation for the sum of $f(i)$ as $i$ goes from 1 to $n$
  - This is a generalization of the addition operator $+$, extending it to apply to a collection of values
  - Each item in that collection is accessed by index
    * e.g., $f(1)$ for index $i=1$, $f(2)$ for index $i=2$, etc.
- The same thing can be done for union $\cup$ or intersection $\cap$

- Exercise: Assume $S_1, S_2, \ldots, S_n$ are finite. What is the size of the (generalized) union of all $S_i$ $[i$ from 1 to $n]$?
  - How about if $S_1, S_2, \ldots, S_n$ are finite and pairwise disjoint?

Exercise

- Claim: For any sets $A$ and $B$, $(A \cap B) \subseteq (A \cup B)$.
  - Prove it, or give a counterexample.
Ordered Pairs and n-tuples

- By design / definition, sets are good for asking questions of membership, but not for questions of relative ordering of elements
- A different structure, an n-tuple, represents elements and their ordering

- Definition: An ordered pair is a pair of elements expressed in parenthesis—e.g., (0,0), (9,17), (Jon Stewart, Stephen Colbert)
  - Order matters, so (17, 9) is not the same as (9, 17)
  - Similarly, order matters, so (0,0) does not contain redundant elements—the 0s are distinct from each other, by position
  - How could we state the criterion for identity for ordered pairs?
- This generalizes to n-tuples with more than 2 elements
  - E.g., (0,0),(3,4,5) are both 3-tuples; (72, 86, 94, 86, 76, 66, 72) is a 7-tuple

Set Product (Cartesian Product)

- Given sets A and B, ordered pairs can represent elements from those sets and which set each element came from
- Definition: The set product (or Cartesian product) of sets A and B is
  \[ A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \} \]
  - That is, it’s the set of all pairs s.t. the first element is in A and the second element is in B
  - Note: This generalizes to more than two sets. \( A_1 \times A_2 \times \ldots \times A_n \) is all n-tuples s.t. the first element is from \( A_1 \), the second from \( A_2 \), …, and the n’th from \( A_n \)
- Exercises
  - What’s \{0, 1\} x \{1, 2, 3\}?
  - What’s \{1,2,3,4,5,6\} x \{1,2,3,4,5,6\}?
  - Let \( A = \{2\} \times \{1, \ldots, 28\} \), \( B = \{4,6,9,11\} \times \{1, \ldots, 30\} \), and \( C = \{1,3,5,7,8,10,12\} \times \{1, \ldots, 31\} \). What is \( A \cup B \cup C \)?

"Cogito ergo product"? Are you sure you don't mean Cartesian sum? Yes, I'm sure.