CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T Th 9:00am
Lab – F 10:30am

Lecture Meeting Location: SP 309
Lab Meeting Location: SP 309

Business

• HW1 due already
• HW2 out today, due Sept. 24
  – Extend to Sept. 29?
  – Note: HW3 will be out on Sept. 24 and due Oct. 8
• Please read Ch.2.1, 2.2.2, 2.3.1, 2.4, and 2.5 in your textbook
  – You can skip section 2.3.2, but people interested in databases might want to read it anyway
• Lab0 due by the end of the day, Thursday, Sept. 17
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach
Ordered Pairs and n-tuples

- By design / definition, sets are good for asking questions of membership, but not for questions of relative ordering of elements
- A different structure, an \textit{n-tuple}, represents elements \textit{and} their ordering

- Definition: An \textit{ordered pair} is a pair of elements expressed in parenthesis—e.g., (0,0), (9,17), (Jon Stewart, Stephen Colbert)
  - Order matters, so (17, 9) is not the same as (9, 17)
  - Similarly, order matters, so (0,0) does not contain redundant elements—the 0s are distinct from each other, by position
  - How could we state the \textit{criterion for identity} for ordered pairs?
- This generalizes to \textit{n-tuples} with more than 2 elements
  - E.g., (0,0),(3,4,5) are both 3-tuples; (72, 86, 94, 76, 66, 72) is a 7-tuple

Set Product (\textit{Cartesian Product})

- Given sets \(A\) and \(B\), ordered pairs can represent elements from those sets \textit{and} which set each element came from
- Definition: The \textit{set product} (or \textit{Cartesian product}) of sets \(A\) and \(B\) is
  \[
  A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}
  \]
  - That is, it’s the set of all pairs s.t. the first element is in \(A\) and the second element is in \(B\)
  - \textbf{Note: This generalizes to more than two sets.} \(A_1 \times A_2 \times \ldots \times A_n\) is all \(n\)-tuples s.t. the first element is from \(A_1\), the second from \(A_2\), …, and the \(n\)'th from \(A_n\)
- Exercises
  - What’s \(\{0, 1\} \times \{1, 2, 3\}\)?
  - What’s \(\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}\)?
  - Let \(A = \{2\} \times \{1,\ldots, 28\}\), \(B = \{4,6,9,11\} \times \{1,\ldots, 30\}\), and \(C = \{1,3,5,7,8,10,12\} \times \{1,\ldots, 31\}\). What is \(A \cup B \cup C\)?
Binary Relations

• A relation, intuitively enough, expresses the relation between elements of various sets. More formally…

• For sets A and B, a binary relation from A to B (or over A x B) is a subset of A x B. (That's the Cartesian product of A and B)
  – i.e., it is a set of ordered pairs of the form (a,b) where a ∈ A and b ∈ B
  – … thus, it relates elements of A to elements of B
  – Note: If a relation R is a subset of A x A, we say it is a relation over A

• More generally, for sets A_1, ..., A_n, an n-place relation over A_1, ..., A_n is a subset of the set product A_1 x … x A_n

Directed Graphs

• One way of representing a binary relation is a directed graph, which indicates ordered relations between elements

(a) and (c) above are directed graphs; (b) is undirected. More on that soon!

• A directed graph G is a pair (V,E) where V is a finite set of vertices (singular: vertex), and E is a binary relation on V x V
  – E is called the edges (or edge set) of graph G

• Exercise: For graph (a), what are the sets V and E?
Graphs, and Inverse of a Relation

• Given a relation $R$, the inverse $R^{-1}$ of $R$ is the set of all ordered pairs $(b,a)$ s.t. $(a,b) \in R$
  – You can think of the inverse as reversing the direction of $R$
  – Your textbook calls this the converse of $R$; I’m more familiar with calling it the inverse

• Questions:
  – Are $R^{-1}$ and $R$ disjoint?
  – What’s the inverse of the relation in (c), below? (express it as ordered pairs, not a graph)

Properties of Relations

• Some useful properties of relations! Definitions:
  – A relation $R$ over a set $A$ is reflexive if for all $a \in A$, $(a,a) \in R$
    • What would a reflexive relation be over the set $\{1,2,3\}$?
    • (If it’s clear in context what the set $A$ is, we might simply say that the relation $R$ is reflexive, in that context)
  – A relation $R$ is symmetric if whenever $(a,b) \in R$, $(b,a) \in R$
  – A relation $R$ is transitive if
    whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

• Exercise: Consider the following relations over $\{1,2,3,4\}$
  • $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
  • $R_2 = \{(1,1), (1,2), (2,1)\}$
  • $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
  • $R_4 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
  – Which of these are reflexive? symmetric? transitive?
More Vocabulary and Properties of Relations

- For relation R:
  - *Domain* $\text{dom}(R) = \{ a \mid \exists b \text{ s.t. } (a,b) \in R \}$ — that is, the domain is all elements that are the first element of a pair in R
  - *Range* $\text{range}(R) = \{ b \mid \exists a \text{ s.t. } (a,b) \in R \}$ — that is, the range is all elements that are the second element of a pair in R

- Recall that relations are sets of n-tuples; binary relations are sets of ordered pairs. So, applying ideas from subsets:
  - A relation R could be a subrelation of a relation S
  - The empty relation has no elements — it is the same as $\emptyset$

- Exercises:
  - What is $\text{dom}(A \times B)$? What is $\text{range}(A \times B)$?
  - What does it mean if relations R and S are disjoint? If $A = \{1,2,3\}$, what is an example of disjoint relations over A?
  - Consider a relation R and its inverse $R^{-1}$. How do $\text{dom}(R)$ and $\text{range}(R)$ relate to $\text{dom}(R^{-1})$ and $\text{range}(R^{-1})$?