CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T Th 9:00am
Lab – F 10:30am

Lecture Meeting Location: SP 309
Lab Meeting Location: SP 309

Business

• HW2 due already
• HW3 out, due October 8
  – (Really, due October 7 / October 8 – see assignment sheet)
  
  Note: Sometimes when I say a HW is due on a certain day, that may mean that the programming part is due the day before—e.g., saying HW3 is due Oct. 8

• Please read Ch.3.1-3.6
• Lab2 due by the end of the day, Thursday, Oct. 1
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach

• Exam: after break
Business, pt. 2

- Google Talk coming soon!
  - **Who:** All Computer Science and Engineering students, but anyone with an interest in software development is welcome!
  - **What:** Your Career at Google. Talk with Luis Ibanez, Software Engineer
  - **Date:** October 3rd
  - **Time:** 1pm
  - **Location:** Taylor Hall 203 Auditorium
  - **RSVP:** https://goo.gl/Qm27Cc

Equivalence Relations

- **Recall definitions:**
  - A relation $R$ over a set $A$ is *reflexive* if for all $a \in A$, $(a,a) \in R$
  - A relation $R$ is *symmetric* if whenever $(a,b) \in R$, $(b,a) \in R$
  - A relation $R$ is *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
- When a relation $R$ has all of these properties, it is called an *equivalence relation*
- If $R$ is an equivalence relation, then it induces *equivalence classes* on the elements
  - For equivalence relation $R$ and any element $a$, let $C_a$ stand for all elements related to element $a$ in $R$—that is, $C_a = \{ b \mid (a,b) \in R \}$
  - Then, $C_a = C_b$ exactly when $(a,b) \in R$!
- **What are some equivalence relations over the natural numbers?**
  - Verify all three properties. What are the equivalence classes?
  - What do the graphs of these equivalence relations look like?
Partitions

More formally, a definition of a partition:

- Let $A$ be a non-empty set, and let $\{B_i\}_{i \in I}$ be an indexed collection of non-empty subsets of $A$ (I is called an index set)

- ... Then, $\{B_i\}_{i \in I}$ is a partition of $A$ iff
  1. $\{B_i\}_{i \in I}$ is a pairwise-disjoint collection (do you remember this definition?)
  2. $\{B_i\}_{i \in I}$ exhausts $A$ (see definition below)

  - Definition: We say that $\{B_i\}$ exhausts $A$ iff $(\bigcup_{\{B_i\}_{i \in I}}) = A$
    - That is, $\forall a \in A, \exists i \in I$ s.t. $a \in B_i$

How does this compare to our intuitive sense(s) of what it means for something to be partitioned?

Equivalence Relations and Partitions

- Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  - Every equivalence relation over $A$ determines a partition over $A$
  - Every partition over $A$ determines an equivalence relation over $A$
  - Thus, in some sense, they're doing the same thing!

- Claim: Every equivalence relation over $A$ determines a partition over $A$
  - Proof:?

- Claim: Every partition over $A$ determines an equivalence relation over $A$
  - Proof:?
Equivalence Relations and Partitions

• Claim: Every partition over \( A \) determines an equivalence relation over \( A \)
  – Proof: Let \( \{B_i\}_{i \in I} \) be a partition of \( A \). Thus, each \( B_i \) is non-empty, and the collection of sets \( \{B_i\}_{i \in I} \) is pairwise disjoint and exhausts \( A \).
  – Define relation \( E = \{(a,b) \mid \exists \ i \text{ s.t. } a, b \text{ are both in } B_i\} \)
    • We say \( E \) is the relation over \( A \) associated with (or induced by) the partition \( \{B_i\}_{i \in I} \).
  – Claim: \( E \) is an equivalence relation over \( A \). Must show \( E \) is…
    1. Reflexive: (Proof?)
    2. Symmetric: (Proof?)
    3. Transitive: (Proof?)

Equivalence Relations and Partitions

• Claim: Every equivalence relation over \( A \) determines a partition over \( A \)
  – Proof: Let \( E \) be an equivalence relation over \( A \).
    Consider the sets \( E_a \) s.t. \( E_a = \{x \mid (a,x) \in E\} \) — i.e., the equivalence classes of \( E \).
  – Then, it remains to prove that the collection of all sets \( \{E_a\} \) are a partition of \( A \). To fit the definition of partition, we must show
    1. Each \( E_a \) is non-empty
    2. The collection \( \{E_a\} \) is pairwise-disjoint
    3. Together, the \( E_a \) exhaust \( A \) (i.e., every element in \( A \) is in at least one \( E_a \)).
(Flighty, but Not A Digression)

- Imagine that a directed graph $G = (V, E)$ represented a flight map, with edges indicating roads (or flights)
  - I.e., for $v, w \in V$, $(v, w) \in E$ iff there’s a direct flight from $v$ to $w$
- So, the graph directly answers the “direct flight” question
  - What other questions might you want to ask?

You might want to ask “What kind of an airline has two flights into Salt Lake City but only one flight into Los Angeles?”

But perhaps there are other questions to ask too?

(“Where Can I Go From Here…?”)

- Imagine that a directed graph represents a flight map, with edges indicating flights (and direction)
  - Edge relation represents, for each vertex $v$ in $V$, what other vertices $w$ were reachable by non-stop flight from $v$
  - I.e., for $v, w \in V$, $(v, w) \in E$ iff there’s a direct flight from $v$ to $w$
- So, the graph directly answers the “direct flight” question
  - What would answer the “where can I go from here, in any number of stops?” question?
  - That is, what would contain all $(v, x)$ if there are any number of flights that could eventually get someone from $v$ to $x$, through any number of stops?

“I’ve been there, I know the way…” — R.E.M.
Transitive Closure

- The transitive closure \( R^* \) of a (binary) relation \( R \) is the unique smallest relation that is transitive and that includes \( R \)
- One way of looking at it:
  - Define \( R^* \) by: \((x_1,x_n)\) is in \( R^* \) iff there is a “path” \( x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_n \) from \( x_1 \) to \( x_n \) in \( R \)—i.e., where every “step” \((x_i, x_j)\) on that path is in \( R \)
  - This is essentially the finite path definition of \( R^* \) on pg. 52 of your textbook
- Another way of looking at it:
  - Top-down definition: \( R^* = \bigcap \{ S : R \subseteq S \subseteq A^2 \text{ and } S \text{ is transitive} \} \)—i.e., \( R \) is the intersection of all possible transitive relations (over \( A \), in this case) that contain \( R \)