CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T Th 9:00am
Lab – F 10:30am

Lecture Meeting Location: SP 309
Lab Meeting Location: SP 309

Business

• HW3 out, due October 8
  – (Really, due October 7 / October 8 – see assignment sheet)
• Please read Ch.3.1-3.6
• Lab2 due by the end of the day, Thursday, Oct. 1
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach
• Change Coaching Hours to have on Thursday?
• The Friday before break
• Exam: after break
Business, pt. 2

• Google Talk coming soon!
  – **Who:** All Computer Science and Engineering students, but anyone with an interest in software development is welcome!
  – **What:** Your Career at Google. Talk with Luis Ibanez, Software Engineer
  – **Date:** October 3rd
  – **Time:** 1pm
  – **Location:** Taylor Hall 203 Auditorium
  – **RSVP:** https://goo.gl/Qm27Cc

Equivalence Relations and Partitions

• Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  – Every equivalence relation over A determines a partition over A
  – Every partition over A determines an equivalence relation over A
  – Thus, in some sense, they're doing the same thing!
• Claim: Every equivalence relation over A determines a partition over A
  – Proof:?
• Claim: Every partition over A determines an equivalence relation over A
  – Proof:?
Equivalence Relations and Partitions

• Claim: Every partition over A determines an equivalence relation over A
  – Proof: Let \( \{B_i\}_{i \in I} \) be a partition of A. Thus, each \( B_i \) is non-empty, and the collection of sets \( \{B_i\}_{i \in I} \) is pairwise disjoint and exhausts A.
  – Define relation \( E = \{(a,b) \mid \exists i \text{ s.t. } a, b \text{ are both in } B_i\} \)
    • We say \( E \) is the relation over A associated with (or induced by) the partition \( \{B_i\}_{i \in I} \)
  – Claim: \( E \) is an equivalence relation over A. Must show \( E \) is...
    1. Reflexive: (Proof?)
    2. Symmetric: (Proof?)
    3. Transitive: (Proof?)

Equivalence Relations and Partitions

• Claim: Every equivalence relation over A determines a partition over A
  – Proof: Let \( E \) be an equivalence relation over A.
    Consider the sets \( E_a \) s.t. \( E_a = \{x \mid (a,x) \in E\} \) — i.e., the equivalence classes of \( E \)
  – Then, it remains to prove that the collection of all sets \( \{E_a\} \) are a partition of A. To fit the definition of partition, we must show
    1. Each \( E_a \) is non-empty
    2. The collection \( \{E_a\} \) is pairwise-disjoint
    3. Together, the \( E_a \) exhaust A (i.e., every element in A is in at least one \( E_a \))
(Flighty, but Not A Digression)

- Imagine that a directed graph \( G = (V,E) \) represented a flight map, with edges indicating roads (or flights)
  - i.e., for \( v, w \in V \), \( (v,w) \in E \) iff there’s a direct flight from \( v \) to \( w \)
- So, the graph directly answers the “direct flight” question
  - What other questions might you want to ask?

You might want to ask “What kind of an airline has two flights into Salt Lake City but only one flight into Los Angeles?”
- But perhaps there are other questions to ask too?

(“Where Can I Go From Here…?”)

- Imagine that a directed graph represents a flight map, with edges indicating flights (and direction)
  - Edge relation represents, for each vertex \( v \) in \( V \), what other vertices \( w \) were reachable by non-stop flight from \( v \)
  - i.e., for \( v, w \in V \), \( (v,w) \in E \) iff there’s a direct flight from \( v \) to \( w \)
- So, the graph directly answers the “direct flight” question
  - What would answer the “where can I go from here, in any number of stops?” question?
  - That is, what would contain all \( (v,x) \) if there are any number of flights that could eventually get someone from \( v \) to \( x \), through any number of stops?

“I’ve been there, I know the way…”
- R.E.M.
Transitive Closure

• The transitive closure $R^*$ of a (binary) relation $R$ is the unique smallest relation that is transitive and that includes $R$

• One way of looking at it:
  - Define $R^*$ by: $(x_1, x_n)$ is in $R^*$ iff there is a “path” $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_n$ from $x_1$ to $x_n$ in $R$—i.e., where every “step” $(x_i, x_j)$ on that path is in $R$
  - This is essentially the finite path definition of $R^*$ on pg. 52 of your textbook

• Another way of looking at it:
  - Top-down definition: $R^* = \bigcap \{S : R \subseteq S \subseteq A^2 \text{ and } S \text{ is transitive}\}$—i.e., $R$ is the intersection of all possible transitive relations (over $A$, in this case) that contain $R$

More Transitive Closure

• The transitive closure $R^*$ of a (binary) relation $R$ is the unique smallest relation that is transitive and that includes $R$

• One more way of looking at it, the “bottom-up” definition:
  Given a relation $R$, consider the following recursive definition of relations $R_i$:
  - $R_0 = R$
  - $R_{n+1} = R_n \cup \{ (a,c) \mid \exists x \text{ s.t. } (a,x) \in R_n \text{ and } (x,c) \in R \}$
  - (What is the relationship between the indices of the $R_i$ and the finite path definition on the previous slide?)

• … then, we define $R^*$ to be the union of all the $R_i$ sets:
  - $R^* = \bigcup \{R_n : n \text{ is a natural number}\}$

• Intuition: Keep adding pairs whenever $(a,x)$ is in the partially constructed $R^*$ relation and $(x,c)$ is in $R$
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• Example: What $R_i$ are constructed to find the transitive closure of relation $R = \{(1,2), (2,3), (3,4)\}$ over $\{1,2,3,4\}$?

(I recommend this definition—it is very useful!)

More Transitive Closure

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• … then, we define $R^*$ to be the union of all the $R_n$ sets:
  - $R^* = \bigcup\{R_n : n \text{ is a natural number}\}$

• Example: What $R_i$ are constructed to find the transitive closure of relation $R = \{(2,3), (2,4), (3,2), (4,1), (4,3)\}$ over $\{1,2,3,4\}$?

(I recommend this definition—it is very useful!)