CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 1:30pm
Lab – F 1:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW2 out, extended deadline Feb. 26 (beginning of class meeting on that day)
• HW1 grading update

• Please read Ch.3.1-3.6
• Lab2 due by the end of the day, Thursday, Feb. 26
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach
Subrelations and Subgraphs

- We can also talk about subgraphs of a graph
  - In general, for a graph $G = (V, E)$, graph $G' = (V', E')$ is a subgraph of $G$ iff $V' \subseteq V$ and $E' \subseteq E$
  - i.e., if the vertices and edges of $G'$ are subsets of the vertices and edges of $G$
  - It can be useful to talk about a subgraph *induced by* a particular subset of $V$—that is the subgraph containing all vertices in $V$ and all edges with both endpoints in $V$

- Exercises:
  - In graph (c) below, what is the subgraph induced by $\{1,2,3\}$? How would you write it as a relation?

Subrelations and Subgraphs II

More exercises:

- In graph (a) below, what is the subgraph/subrelation induced by $\{1,2,4\}$?
  - What properties of relations does that subrelation satisfy?
- In graph (a) below, what is the subgraph/subrelation induced by $\{4,5\}$?
  - What properties of relations does that subrelation satisfy?
- In graph (a) below, what is the subgraph/subrelation induced by $\{2\}$?
  - What is the subgraph/subrelation induced by $\{4\}$?
  - What properties of relations do those subrelations satisfy?
Other Properties of Relations

- Recall definitions:
  - A relation $R$ over a set $A$ is **reflexive** if for all $a \in A$, $(a,a) \in R$
  - A relation $R$ is **symmetric** if whenever $(a,b) \in R$, $(b,a) \in R$
  - A relation $R$ is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
- Relations can also be described as, in some sense (**be very careful!**), having the opposites of those properties (**note the $\notin$ symbols!**):
  - A relation $R$ over a set $A$ is **irreflexive** if for all $a \in A$, $(a,a) \notin R$
  - A relation $R$ is **asymmetric** if whenever $(a,b) \in R$, $(b,a) \notin R$
  - A relation $R$ is **intransitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \notin R$
- Note that, e.g., not being reflexive is **not the same** as being irreflexive!
- Exercises:
  - Give a relation over $\{1,2,3\}$ that is neither reflexive nor irreflexive
  - Give a relation over $\{1,2,3\}$ that is neither symmetric nor asymmetric
  - Give a relation over $\{1,2,3\}$ that is neither transitive nor intransitive

Equivalence Relations

- Recall definitions:
  - A relation $R$ over a set $A$ is **reflexive** if for all $a \in A$, $(a,a) \in R$
  - A relation $R$ is **symmetric** if whenever $(a,b) \in R$, $(b,a) \in R$
  - A relation $R$ is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
- When a relation $R$ has all of these properties, it is called an **equivalence relation**
- If $R$ is an equivalence relation, then it induces **equivalence classes** on the elements
  - For equivalence relation $R$ and any element $a$, let $C_a$ stand for all elements related to element $a$ in $R$—that is, $C_a = \{b \mid (a,b) \in R\}$
  - Then, $C_a = C_b$ exactly when $(a,b) \in R$!
- What are some equivalence relations over the natural numbers?
  - Verify all three properties. What are the equivalence classes?
  - What do the graphs of these equivalence relations look like?
Partitions

• Intuitively, what does it mean to you if something—or a collection of things—is *partitioned*?

• How could we write that definition in rigorous, formal notation?

Computer scientists can often do this kind of thing—studying what is meant by something and coming up with a rigorous, formal definition that fits its applications. That way, even a computer can work with that definition!

Partitions

• Intuitively, a *partition* of a set \( S \) is a way of breaking \( S \) into a collection of non-empty subsets \( S_1, S_2, \ldots, S_n \) such that
  – the subsets include all of \( S \);
  – … and the subsets don’t overlap.

• Below, a partition of the set \( S = \{0, 1, \ldots, 10\} \)

• What’s the connection between a *partition* and an *equivalence relation*?
Partitions

More formally, a definition of a partition:

- Let \( A \) be a non-empty set, and let \( \{B_i\}_{i \in I} \) be an indexed collection of non-empty subsets of \( A \) \((I \) is called an index set) \)

- ... Then, \( \{B_i\}_{i \in I} \) is a partition of \( A \) iff
  1. \( \{B_i\}_{i \in I} \) is a pairwise-disjoint collection (do you remember this definition?)
  2. \( \{B_i\}_{i \in I} \) exhausts \( A \) (see definition below)

  - Definition: We say that \( \{B_i\}_i \) exhausts \( A \) iff \( (\bigcup \{B_i\}_{i \in I}) = A \)
    - That is, \( \forall \ a \in A, \exists i \in I \ s.t. \ a \in B_i \)

How does this compare to our intuitive sense(s) of what it means for something to be partitioned?

Equivalence Relations and Partitions

- Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  - Every equivalence relation over \( A \) determines a partition over \( A \)
  - Every partition over \( A \) determines an equivalence relation over \( A \)
  - Thus, in some sense, they're doing the same thing!

- Claim: Every equivalence relation over \( A \) determines a partition over \( A \)
  - Proof:?

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  - Proof:?