CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 1:30pm
Lab – F 1:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW4 out, due April 5 & April 6 (see HW sheet for details)
  – (Extension on those due dates?)

• Reading: Makinson, Ch.4.1-4.6
  – Our coverage of the material will be different from that in the textbook, but it’s good to see the textbook’s presentation, as well

• Reading: Prof. Hunsberger’s document “The Natural Numbers, Induction, and Numeric Recursion”
  – Posted on the Additional Notes / Readings page of the CS145 website
Recursive (or Inductive) Definition of a Set

• A recursive definition of a set $S$ consists of three components:
  – **Base**: One or more foundational elements of $S$
  – **Induction**: One or more rules to construct new elements of $S$ from existing elements of $S$
  – **Closure**: The condition that $S$ consists of all and exactly the elements derived from the base elements and induction rules.

  (In the context of a definition, this is often assumed rather than explicitly stated—the fact that it is a definition means that the set is exactly the elements thus specified.)

• What are some recursively defined sets we’ve already seen, and what are their definitions?

A Recursively Defined Set: The Natural Numbers

• As suggested by all of our inductive proofs about numbers, there is also a recursive definition of the natural numbers
• The **Peano axioms** are conventionally taken as a definition of the natural numbers (here, let $N$ stand for the natural numbers):
  1. There exists a number 0 s.t. $0 \in N$
  2. Every natural number $n$ has a natural number successor, denoted by $S(n)$
  3. There is no $n$ in $N$ s.t. $S(n) = 0$
  4. Distinct natural numbers have distinct successors: if $a \neq b$, then $S(a) \neq S(b)$
  5. Let $P$ be a property of the natural numbers such that:
     • $P(0)$ holds
     • For every $a$ in $N$, if $P(a)$ holds, then $P(S(a))$ holds

Axioms 1 and 2 give the recursive construction of the elements of $N$. The other Axioms are properties of $N$. Axiom 5 is sometimes called the Axiom of Induction.
Peano Examples

• The Peano axioms:
  1. There exists a number 0 s.t. 0 ∈ N
  2. Every natural number has a natural number successor, denoted by S(n)
  3. There is no n in N s.t. S(n) = 0
  4. Distinct natural numbers have distinct successors: if a ≠ b, then S(a) ≠ S(b)
  5. Let P be a property of the natural numbers such that:
     • P(0) holds
     • For every a in N, if P(a) holds, then P(S(a)) holds
   If both of those conditions are true, then P(n) holds for all n in N.

• Exercises and examples:
  – How would we write the number 2 in this notation? The number 5?
  – Could we write the number -1 in this notation?

Peano Examples

• The Peano axioms:
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     • For every a in N, if P(a) holds, then P(S(a)) holds
   If both of those conditions are true, then P(n) holds for all n in N.

• Exercises and examples:
  – Are there two different ways to write the number 2? (Hint: Prove the following Theorem)
  – Proof: By induction!

• Theorem: Every natural number n has one of two forms, either n=0 or n=S(m) for some m—these are the only cases!

(See Hunsberger’s “The Natural Numbers…” document)
Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…
  – What would the base case(s) be?
  – What would the inductive case(s) be?

Note: We proved that every natural number is either 0 or S(m) for some m. How can that help us in this recursive definition?

More Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…

• Definition of addition:
  1. Case z=0 -- For all n in N, n + 0 = n
  2. Case z=S(m) -- For all n in N, z = S(m) for some m in N:
     n + S(m) = S(n + m)

• Let’s prove something with that definition!
  – Claim: This addition function is associative (i.e., a + (b + c) = (a + b) + c, for all a,b,c in N)
  – Proof: ??