CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 1:30pm
Lab – F 1:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW7: Full HW out today
  – Due May 8 / May 9 (see HW sheet)
  – (We may not have covered all the probability material for some exercises by the end of class today, but it’s coming soon!)
• Reading: Makinson, Ch. 6
  – We may not cover all the material, but it’s worth reading anyway
• HW5 back today

• Very probably lecture instead of lab on Friday
  – I’ll email to confirm
Probability Function

- A probability distribution assigns a probability (in [0, 1]) to each single element of S (i.e., each elementary event)
- A probability function is an extension of a probability distribution, so it applies to all events—i.e., all subsets of S—not just all elements of S
  - Probability function $p^*: P(S) \rightarrow [0,1]$ assigns values to elements of the power set of S
  - $p^*(A) = \sum\{p(a) \mid a \in A\}$ for non-empty subsets A of S
  - $p^*(\emptyset) = 0$
- Terminology
  - The pair of a sample space S and a probability function $p^*$ over S, $(S, p^*)$, is called a probability space

Probability Spaces

- Terminology
  - The pair of a sample space S and a probability function $p^*$ over S, $(S, p^*)$, is called a probability space
- For this example, what is the relevant sample space? And what are the relevant probabilities?
  - Select a playing card at random from a standard deck. What is the probability that it is…
    - A diamond? (Consider event D: a card is a diamond. What’s $p(D)$)
    - A face card? (Consider event F: a card is a face card. What’s $p(F)$)
    - Both? Either? (What events do we consider for these cases?)
    - A diamond but not a face card? (What event do we consider?)
Probability Spaces

- Terminology
  - The pair of a sample space $S$ and a probability function $p^*$ over $S$, $(S, p^*)$, is called a probability space.
- For this example, what is the relevant sample space? And what are the relevant probabilities?
  - Toss a fair coin 3 times. What is the probability that…
    - at least two consecutive tosses are heads? (What event should we consider?)
    - exactly two of the tosses (not necessarily consecutive) are heads?

Conditional Probability, pt. 1

- Sometimes, there are questions of the form “What is the probability of <something>, given that <something else>?”
  - Example: You roll a pair of fair dice. What is the probability that at least one of the dice is showing a 3, given that their sum is 5?
  - How would you approach this example?
Conditional Probability, pt. 1

- Sometimes, there are questions of the form “What is the probability of <something>, given that <something else>?"  
  - Example: You roll a pair of fair dice. What is the probability that at least one of the dice is showing a 3, given that their sum is 5?  
  - We might think the sample space is all rolls of two dice:  
    \[ S = \{(i, j) \mid i, j \in \{1, 2, 3, 4, 5, 6\}\} \]  
  - But that may not really be the sample space here… the example isn’t really asking about considering all rolls  
  - Instead, we might think the sample space is the set of all rolls that sum to 5:  
    \[ \{(i, j) \mid i + j = 5 \text{ and } i, j \in \{1, 2, 3, 4, 5, 6\}\} \]  
  - Then, from that sample space, could consider the event that at least one of the dice shows a 3

Conditional Probability, pt. 2

- Changing sample spaces can sometimes be inconvenient or counterintuitive  
- It can be more convenient to keep the same sample space and instead get the probability in a different manner  
  - Let \( S \) be our sample space: all rolls of two dice  
  - Let \( A \) be the event that at least one of the dice shows a 3  
  - Let \( B \) be the event that the sum of the dice is 5  
- We can then consider the conditional probability of \( A \) given \( B \)  
  - (i.e., the probability of at least one of the dice showing 3, given that their sum is 5)  
  - Notation: We can write that probability as \( p(A \mid B) \)
Conditional Probability, pt. 2

• Our example:
  – Let S be our sample space: all rolls of two dice
  – Let A be the event that at least one of the dice shows a 3
  – Let B be the event that the sum of the dice is 5
• Consider the conditional probability of A given B
  – (i.e., the probability of at least one of the dice showing 3, given that their sum is 5)
  – Notation: We can write that probability as \( p(A | B) \)
  – Calculation: \( p(A | B) = \frac{p(A \cap B)}{p(B)} \)
  – That is, the probability of A given B is the probability of A and B divided by the probability of B alone
• What is the prior (un-conditional) probability \( p(A) \)?

Conditional Probability, pt. 2

• Our example:
  – Let S be our sample space: all rolls of two dice
  – Let A be the event that at least one of the dice shows a 3
  – Let B be the event that the sum of the dice is 5
• Consider the conditional probability of A given B
  – Calculation: \( p(A | B) = \frac{p(A \cap B)}{p(B)} \)
• In this case, what is the conditional probability that the sum of the dice is 5 given that at least one of the dice shows a 3
  – That is, what is \( p(B | A) \)?
  – Is it going to be the same as \( P(A | B) \)?
  – How do we compute it?
More Conditional Examples

- In a writer’s group…
  - 60% of the members write novels
  - 30% of the members write poetry
  - 10% of the members write both novels and poetry
  - 15% of the members write songs
  - All of the songwriters write poetry
  - 80% of the songwriters write love poetry
  - None of the novelists write love poetry

- What is the probability that:
  - An arbitrarily chosen poet writes novels?
  - An arbitrarily chosen novelist writes poetry?

More Conditional Examples

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- What is the probability that:
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  - An arbitrarily chosen songwriter writes novels?
Two Definitions of Independence, pt. 1

- The notion of events being independent arises frequently in probability
  - i.e., when a fair coin is flipped, its landing Heads or Tails is taken to be independent of the outcomes of any other flips of that coin
- Independence can be defined in terms of probabilities:

- Let $p$ be a probability function on a (finite) sample space $S$, and let $A, B$ be events.
- Then, we say $A$ is independent of $B$ iff $p(A) = p(A \mid B)$
  - ... or $p(B) = 0$

Note: This notion of probabilistic independence is "not a matter of the presence or absence of a causal connection" (Makinson, pg. 155)

Two Definitions of Independence, The Sequel—Still Independent

- Let $p$ be a probability function on a (finite) sample space $S$, and let $A, B$ be events.
  - Then, we say $A$ is independent of $B$ iff $p(A) = p(A \mid B)$
  - ... or $p(B) = 0$
- Alternatively, we say $A$ is independent of $B$ iff
  \[ p(A \cap B) = p(A) \times p(B) \]
- Are these definitions equivalent?
- A note about complements:
  1. $A$ and $B$ are independent iff...
  2. $A$ and $\neg B$ are independent iff...
  3. $\neg A$ and $B$ are independent iff...
  4. $\neg A$ and $\neg B$ are independent

Hint: Yes they are equivalent. I'll leave it to you to go through the proof as an exercise... please come to me if you have questions on it!

That is, the conditions numbered 1-4 here are all equivalent. See Ch. 6.6 in your book for more info!