CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 10:30am
Lab – F 3:10pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• Reading: Ch. 1.1-1.4 in our textbook
• See new link from course webpage Additional Notes / Readings for a document about induction
• HW1 out today, due Wednesday February 10
  – (available at course website)
• Questions about the last lecture? (Or the reading?)
• Coaching hours still TBD
  – Temp. Coaching Hours: Monday, Tuesday this week, 8-10pm
• Please email me (as per Assignment from first lecture)
  – Note: my preferred email is eaaron@cs.vassar.edu
    (not eraaron@vassar.edu)
  – Thank you to those who have emailed me already!
Mathematical In(tro)duction, cont.

• Intuitive idea: Proving that all the dominos will fall
  (1) First, make sure the first domino falls
  (2) Then, make sure they’re all set up such that if you look at any domino in the chain—say, the k’th domino, for any k—then if the k’th domino falls, then the (k+1)’st will fall

• This is enough to prove that every domino falls!

  Think about it a bit…
  – The first one falls, because we prove that directly with (1)
  – The second one falls, because we consider (2) with k = 1—it says if the first domino falls, so does the second
  – Similarly, the third one falls, because we consider (2) with k=2…
  – Similarly, the fourth one falls… (is it clear why?)
  – And we can show that all of them fall! How?

Inductive Proofs

• Our inductive proof, like all inductive proofs, has four parts
  (1) Write down what we’re trying to prove
     • Be sure to describe the variables! (e.g., “For every number n greater than 1…”)
  (2) Prove the base case—the first domino falls
  (3) Write the inductive hypothesis—assume domino k falls
  (4) Prove your inductive case—using the inductive hypothesis, show that domino k+1 falls too

Every step in this can be subtle, or require some thought—we’ll see examples as the course goes on!
Another Example Induction

- Show that the sum of the first $n$ odd numbers is $n^2$
- Proof by induction! Go through the steps
  - Write down what we’re trying to prove
  - What’s our base case?
  - What’s our inductive hypothesis?
  - How do we use our inductive hypothesis to prove our inductive case?

Getting started with induction can feel a little getting started with recursion—like there’s some magic to it! (“Wait… that’s the whole recursive program?”)

But as with recursion, you learn induction by thinking through several examples. We’ll see more of it as the course goes along!

The Set Up!

- Sets are one of the most important structures / concepts for the mathematics of CS
- Definition (intuitive): A set is a collection of distinct objects
  - Each entity / object is called an element of the set
  - Primary operation on sets: membership—i.e., for element $x$ and set $A$, if $x$ is an element of $A$, we write $x \in A$
  - (if $x$ and $y$ are both elements of $A$, we can write $x, y \in A$)
  - … and if $x$ is not an element of $A$, we write $x \not\in A$

Typically, a set is written as a Capital letter, and a non-set element is lowercase

- Note: Any kind of object can be an element of a set—a number, a person, a song,… or even another set!
Set Up (cont.)

- Sets are written as elements in curly braces
  - e.g., \{1, 2, 3, 4, 5\}, \{\{CS145\}, \{CS240, CS241\}\}, \{“Some Nights”, “We Are Young”, “Carry On”\}, etc.
  - Be very careful using ellipsis dots (…) in writing
    a set: e.g., \{1, 2, 3, 4, 5, …\}—is that all positive integers? all positive integers less than 100? etc.

- Some important conventions / properties:
  - Order doesn’t matter: \{5, 9\} is the same as \{9, 5\}
  - Repeated elements are not considered: \{5, 5, 9\} is the same as \{5, 9, 9\} is the same as \{5, 9\}

  - Do these conventions / properties make sense to you?
  - How do they relate to the idea that membership is the primary operation on sets?

Set Equality (the Identity relation)

- Definition: Two sets A, B are said to be equal (or identical) if they contain exactly the same elements
  - So, \{5, 9\} = \{9, 5\} = \{5, 5, 9\} (as noted before)

Important note:

It is often confusing (or “inelegant”, as your textbook puts it) to explicitly write an element twice, as in \{5, 5, 9\}.

Please use good style—avoid needless inelegance and minimize confusion in your written work for this course! Excessively inelegant or confusing answers to exercises may not receive full credit.

If there are questions on this anytime in the semester, please ask me!
Subset (the *Inclusion* relation)

- **Definition:** Set A is said to be a *subset* of set B if every element of A is also an element of B
  - *i.e.,* iff (“if and only if”) for all x, if \( x \in A \) then \( x \in B \)
- **Terminology / notation:**
  - We write \( A \subseteq B \) to represent that A is a subset of (and possibly equal to!) B
  - When A is a subset of B, we can also say B is a *superset* of A, or B *includes* A, written as \( B \supseteq A \)

  *Your textbook (see Ch. 1.2, pg. 2) draws a distinction between the word *include* and the word *contain* in this context. Please be aware of that distinction. Also, please be aware that it is not always conventionally followed outside of your textbook!*

Subset (cont.)

- **More terminology / notation:**
  - Recall, notation \( A \subseteq B \) indicates that A is a subset of B (and possibly equal to B)—the horizontal line looks like part of an = sign
  - Notation \( A \subsetneq B \) explicitly indicates that A is a subset of B and *not* equal to B
  - In the case that \( A \subsetneq B \), we say A is a *proper subset* of B

  *In your textbook, notation \( A \subset B \) also indicates that A is a proper subset of B, but …
  *… in other sources, sometimes \( A \subset B \) indicates that A is a subset of B, but doesn’t say anything about whether or not they are equal
  *For now, our default will be the convention of your textbook (which your Prof. will try not to forget!)*

- **Exercises:**
  - The notation \( A \not\subset B \) indicates …? How about \( A \not\subseteq B \)?
Subset and Membership, cont.

• Membership / inclusion symbols can exist in both directions, and with negated forms, such as:
  – for membership: ∈, ∉, ∋, ∌
  – for inclusion: ⊂, ⊃, ⊄, ⊉, ⊆, ⊇, ⊈, ⊉

• If A ⊆ B and B ⊆ A, what else can we say about a relationship between sets A and B?

• Questions / exercises:
  – For a set A, is it necessarily true that A ⊆ A?
  – How about A ⊂ A? Or A ⊇ A? Or A ∈ A?

Exercises

• Which sets are included in which?
  – A) set of positive integers less than 10
  – B) set of prime numbers less than 11
  – C) set of odd numbers between 1 and 6 (exclusive)
  – D) {1, 2}
  – E) {1}
  – F) set of prime numbers less than 8

• Which of those sets are identical to which?
Exercises

• True or false?
  (a) Whenever $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$
  (b) Whenever $A \subseteq B$ and $C \subseteq B$, $A \subseteq C$
  (c) Whenever $A_1 \subseteq A_2 \subseteq A_3 \ldots \subseteq A_n$, and also
      $A_n \subseteq A_1$, then $A_i = A_j$ for all $i, j$ in $[1..n]$
  (d) $A = B$ iff neither $A \subset B$ nor $B \subset A$
  (e) Whenever $A \subset B$ and $B \subseteq C$, $A \subset C$
Intersection

- There are some fundamental set operations, i.e., ways of constructing sets from other sets.
- Example: Intersection
  - Intuitively, an intersection is what two (or more) things have in common.
  - For sets A, B, the intersection $A \cap B$ is the set of elements that A and B have in common. More formally...
- Definition: The intersection $A \cap B$ of sets A and B is defined by: $x \in A \cap B$ iff $x \in A$ and $x \in B$.
  - Also written as $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- Examples:
  - What’s $\{1, 2, 3, 4\} \cap \{2, 4, 6, 8\}$?
  - What’s $\{x \mid x \text{ is odd between 0 and 10}\} \cap \{x \mid x \text{ is prime between 0 and 10}\}$.

Union

- Another set operation: Union
  - Intuitively, a union is the combination of what two (or more) things have when they’re put together.
  - For sets A, B, the union $A \cup B$ is the set of elements that either A or B contain. (Or both! This is a non-exclusive or.) More formally...
- Definition: The union $A \cup B$ of sets A and B is defined by: $x \in A \cup B$ iff $x \in A \text{ or } x \in B$.
  - Also written as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- Examples:
  - What’s $\{1, 2, 3, 4\} \cup \{2, 4, 6, 8\}$?
  - What’s $\{x \mid x \text{ is odd between 0 and 10}\} \cup \{x \mid x \text{ is prime between 0 and 10}\}$. 