CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 10:30am
Lab – F 3:10pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW3 out
  – HW3 due March 8 / March 9 (see assignment sheet)

  Note: Sometimes when I say a HW is due on a certain day, that may mean that
  the programming part is due the day before—e.g., saying HW3 is due March 9

• Please read Ch.3.1-3.6
• Lab3 due by the end of the day, Thursday, March 3
  – Must be submitted online (using the submit145 command)
  – Must be checked off by me or a Coach
Equivalence Relations and Partitions

• Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  – Every equivalence relation over A determines a partition over A
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  – Thus, in some sense, they're doing the same thing!

• Claim: Every equivalence relation over A determines a partition over A
  – Proof:?

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Equivalence Relations and Partitions

• Claim: Every equivalence relation over A determines a partition over A
  – Proof: Let E be an equivalence relation over A.
    Consider the sets $E_a$ s.t. $E_a = \{ x \mid (a, x) \in E \}$—i.e., the equivalence classes of E
  – Then, it remains to prove that the collection of all sets $\{E_a\}$ are a partition of A. To fit the definition of partition, we must show
    1. Each $E_a$ is non-empty
    2. The collection $\{E_a\}$ is pairwise-disjoint
    3. Together, the $E_a$ exhaust A (i.e., every element in A is in at least one $E_a$)
(Flighty, but Not A Digression)

- Imagine that a directed graph $G = (V,E)$ represented a flight map, with edges indicating roads (or flights)
  - I.e., for $v, w \in V$, $(v,w) \in E$ iff there’s a direct flight from $v$ to $w$
- So, the graph directly answers the “direct flight” question
  - What other questions might you want to ask?

You might want to ask “What kind of an airline has two flights into Salt Lake City but only one flight into Los Angeles?”

But perhaps there are other questions to ask too?

(“Where Can I Go From Here…?”)

- Imagine that a directed graph represents a flight map, with edges indicating flights (and direction)
  - Edge relation represents, for each vertex $v$ in $V$, what other vertices $w$ were reachable by non-stop flight from $v$
  - I.e., for $v, w \in V$, $(v,w) \in E$ iff there’s a direct flight from $v$ to $w$
- So, the graph directly answers the “direct flight” question
  - What would answer the “where can I go from here, in any number of stops?” question?
  - That is, what would contain all $(v,x)$ if there are any number of flights that could eventually get someone from $v$ to $x$, through any number of stops?

“I’ve been there, I know the way…”

– R.E.M.
Transitive Closure

• The transitive closure $R^*$ of a (binary) relation $R$ is the unique smallest relation that is transitive and that includes $R$

• One way of looking at it:
  – Define $R^*$ by: $(x_1, x_n)$ is in $R^*$ iff there is a “path” $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_n$ from $x_1$ to $x_n$ in $R$—i.e., where every “step” $(x_i, x_j)$ on that path is in $R$
  – This is essentially the finite path definition of $R^*$ on pg. 52 of your textbook

• Another way of looking at it:
  – Top-down definition: $R^* = \cap\{S : R \subseteq S \subseteq A^2$ and $S$ is transitive\}—i.e., $R$ is the intersection of all possible transitive relations (over $A$, in this case) that contain $R$

More Transitive Closure

• The transitive closure $R^*$ of a (binary) relation $R$ is the unique smallest relation that is transitive and that includes $R$

• One more way of looking at it, the “bottom-up” definition:
  Given a relation $R$, consider the following recursive definition of relations $R_i$:
  – $R_0 = R$
  – $R_{i+1} = R_i \cup \{(a,c) \mid \exists x \text{ s.t. } (a,x) \in R_i \text{ and } (x,c) \in R\}$
  – (What is the relationship between the indices of the $R_i$ and the finite path definition on the previous slide?)

• … then, we define $R^*$ to be the union of all the $R_i$ sets:
  – $R^* = \bigcup\{R_i : n \text{ is a natural number}\}$

• Intuition: Keep adding pairs whenever $(a,x)$ is in the partially constructed $R^*$ relation and $(x,c)$ is in $R$
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• Example: What $R_i$ are constructed to find the transitive closure of relation $R = \{(1,2), (2,3), (3,4)\}$ over $\{1,2,3,4\}$?

(I recommend this definition—it is very useful!)

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