CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 10:30am
Lab – F 3:10pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW3 extended deadline: March 10 / March 11 (instead of March 8 / 9 on assignment sheet), as emailed
• HW2 back today
• Please read Ch.3.1-3.6

• Friday, March 11
  – Review session; not a lab
• Monday, March 28
  – Review session
• Wednesday, March 30
  – Exam: Will cover HWs 1-3
  – HW3 will be returned and reviewed as part of a review session on the first Monday after break
Relations and Functions

- A (binary) relation is a very general thing—it relates elements to other elements.
- A function is really just a specific kind of relation.
  - You’ve worked with functions before, in the mathematical sense… things like \( f(x) = x^2 \) or \( g(x,y) = x + y \).
  - Function relate things to other things, too—they’re a kind of relation.
- For a one-place (or unary) function (i.e., a function of one argument):
  - Definition: A one-place function from a set A into a set B is any binary relation R from A to B s.t. for all \( a \) in A there is exactly one \( b \) in B for which \( (a,b) \in R \).

Composition of Functions

- Can you think of an application where you want to call one function \( g \) on the results of another function \( f \)?
  - … That is, where the argument (or input) to \( f \) results in a value (or output) from \( f \), and that output is used as the input to \( g \), to get output from \( g \)?
- This is extremely important in Computer Science, and it’s referred to as a *composition of functions*.
  - For function \( f: A \rightarrow B \) and function \( g: B \rightarrow C \), function \( g \circ f \) is the composition of \( g \) and \( f \).
  - \( g \circ f \) is a function from \( A \) to \( C \), where \( (g \circ f)(a) = g(f(a)) \).
  - How do we know \( g \circ f \) is a function?

Note: When proving something is a function, show it is type correct, as well—here, we must show \( g \circ f \) takes inputs from \( A \) and gives outputs in \( C \) as part of a proof!
Example Compositions, to learn from

(Like, etudes?)

- **A composition of functions**
  - For function $f: A \rightarrow B$ and function $g: B \rightarrow C$, function $g \circ f$ is the composition of $g$ and $f$
  - $g \circ f$ is a function from $A$ to $C$, where $(g \circ f)(a) = g(f(a))$

- For these functions $f$ and $g$, what is $g \circ f$? What is $f \circ g$?
  - $f(n) = n + 3; g(n) = 3n$
  - $f(n) = (n + 1)^2; g(n) = n - 1$
  - $f(n) = n + 5; g(n) = n - 5$
  - $f = \{(1,1), (2,2), (3,1)\}$, $g = \{(1,2), (2,3), (3,2)\}$ (both over $\{1,2,3\}$)

Function Inverses

- Recall for relations that an inverse $R^{-1}$ of a relation $R$ was defined as the set of all ordered pairs $(b,a)$ s.t. $(a,b) \in R$—i.e., $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

- For functions, the concept of an inverse is the same—after all, a function is just a relation—but...
  - We know that the inverse of a relation is a relation
  - Is the inverse of a function always a function?

- Consider some examples. Let $A = \{1,2,3\}$, $B = \{a,b,c,d\}$
  - $f = \{(1,a), (2,d), (3,c)\}$. Is $f^{-1}$ a function from $B$ to $A$? Is it a function with some other domain and range?
  - $f = \{(1,a), (2,b), (3,c)\}$. Is $f^{-1}$ a function from $B$ to $A$? Is it a function with some other domain and range?
  - For this $A$ and $B$, could there ever be a function from $A$ to $B$ for which $f^{-1}$ is a function from $B$ to $A$? Why or why not?
Even More Function Words: Injective, Surjective, Bijective

For a function f: A → B

- We say f: A → B is injective (or one-to-one) iff
  for any x, y ∈ A, whenever x ≠ y, f(x) ≠ f(y)
- We say f: A → B is surjective (or onto) iff
  for all b ∈ B, there exists some a ∈ A s.t. f(a) = b
    - Could there be more than one a ∈ A s.t. f(a) = b?
    - Could there be more than one b ∈ B s.t. f(a) = b?
- We say f: A → B is bijective iff it is one-to-one and onto
  - If a function is bijective, we can say it is a one-to-one correspondence between A and B
  - Why does it make sense to call it a one-to-one correspondence?
    Did we just say that about injective functions?

Hint: No we didn’t, though it kind of looks like we did. And the difference is important.

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- We say f: A → B is bijective iff it is one-to-one and onto
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In each of those cases (injective, surjective, bijective), what can we say about the inverse f⁻¹?

- Is f⁻¹ necessarily a function?
- If so, what is its domain and range?
- If not, why not?
Intro to Cardinality Principles

- Intuition: If a relation from A to B is many-to-one, that *seems* to say something about the relative sizes of its *source* A and *target* B
- … but relations are too general to support that intuition the way we want.
- Functions, however, do let us make statements about the relative sizes (or *cardinalities*) of sets
  - Notation: For any finite set S, let |S| stand for the size of set S (i.e., the number of elements in S)
  - Your textbook uses the notation #(S) for the size of S

Cardinality Principles:
The Principle of Equinumerosity

- Functions, however, do let us make statements about the relative sizes (or *cardinalities*) of sets
- *The Principle of Equinumerosity*: For finite sets A and B, 
  |A| = |B| iff there exists a bijection f from A to B
  - Recall f: A → B is *injective* (or *one-to-one*) iff
    for any x, y ∈ A, whenever x ≠ y, f(x) ≠ f(y)
  - Recall f: A → B is *surjective* (or *onto*) iff
    for all b ∈ B, there exists some a ∈ A s.t. f(a) = b
  - Recall f: A → B is *bijective* iff it is one-to-one and onto
- Does this Principle make sense? How could we prove it?
  - Hint: Put all the elements in A in an indexed list. Do the same for B. What could we use for function f that would be a bijection?
Cardinality Principles: The Principle of Comparison

- *The Principle of Equinumerosity*: For finite sets $A$ and $B$, $|A| = |B|$ iff there exists a bijection $f$ from $A$ to $B$

- *The Principle of Comparison*: For finite sets $A$ and $B$, $|A| \leq |B|$ iff there exists an injection (injective function) from $A$ to $B$
  - Recall $f: A \rightarrow B$ is *injective* (or *one-to-one*) iff 
    for any $x, y \in A$, whenever $x \neq y$, $f(x) \neq f(y)$
  - Recall $f: A \rightarrow B$ is *surjective* (or *onto*) iff 
    for all $b \in B$, there exists some $a \in A$ s.t. $f(a) = b$
  - Recall $f: A \rightarrow B$ is *bijective* iff it is one-to-one and onto

- Does this Principle make sense? How could we prove it?
  - Hint: Put all the elements in $A$ in an indexed list. Do the same for $B$. What could we use for function $f$ that would be injective?