CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – M W 10:30am
Lab – F 3:10pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW5 out, due April 19 / April 20 (see HW sheet)

• Lecture, not lab, in Friday’s class this week

• Exam grading update

• Reading: Makinson, Ch.4.1-4.6
  – Our coverage of the material will be different from that in the textbook, but it’s good to see the textbook’s presentation, as well

• Also Reading: Makinson, Ch. 5
Sometimes One Just Isn’t Enough:
The Strong Induction Story

• **Strong induction** is an alternative to standard induction
  – If the base case is some number \( b \) (i.e., Base case: show \( P(b) \) is true)
  – Then in strong induction, the I.H. is Assume \( P(n) \) for all \( b \leq n < k \)
  – And the inductive case is Show \( P(k) \)
  – Do you see how this follows from the same principles as standard induction? Think of dominoes falling….

• Example:
  – Claim: Every natural number \( n \geq 2 \) is either prime or the product of prime numbers.
  – Proof: **By strong induction!** Let \( P(n) \) be “\( n \) is either prime or the product of primes”.
  – What’s the base case? \( n = 2 \); prove \( P(2) \). Proof: \( 2 \) is prime.
  – Inductive hypothesis: Assume \( P(n) \) for all \( 2 \leq n < k \)
  – Inductive case: Show \( P(k) \). What’s the proof?

Postage Stamps

• Induction can be subtle
• Try proving the following claim by induction:
• Claim: Any postage cost of 4 or more cents can be exactly covered by 2-cent and 5-cent stamps
  – Example: 23 cents is \( 4 \) 2-cent stamps and \( 3 \) 5-cent stamps.
  – So, consider \( P(n) = “n = 2*x + 5*y \) for some natural numbers \( x, y \)”
  – Base case: \( n=4 \), Show \( P(4) \).
  – Inductive case: Assume \( P(k) \), show \( P(k+1) \) …
  \textbf{For arbitrarily chosen} \( k \)—that is, it must be proved for all values of \( k \)
Postage Stamps

• Induction can be subtle
• Try proving the following claim by induction:
• Claim: Any postage cost of 4 or more cents can be exactly covered by 2-cent and 5-cent stamps
  – Example: 23 cents is 4 2-cent stamps and 3 5-cent stamps.
  That is, $23 = 4 \cdot 2 + 3 \cdot 5$.
  – So, consider $P(n) = "n = 2x + 5y"$ for some natural numbers $x, y$.
  – Base case: $n=4$. Show $P(4)$.
  – Inductive case: Assume $P(k)$ for arbitrary $k \geq 4$, show $P(k+1)$.
    • By I.H., $k = 2x + 5y$ for some $x, y$. Then, show there exist $x', y'$ such that $(k+1) = 2x' + 5y'$:
      • $k+1 = 2x + 5y + 1 = 2x + 5(y-1) + 6 = 2(x+3) + 5(y-1)$. We’re done, right?
        [Hint: No, we’re not done. What’s the error in this proof?]

Postage Stamps

• Claim: Any postage cost of 4 or more cents can be exactly covered by 2-cent and 5-cent stamps
  – Consider $P(n) = "n = 2x + 5y"$ for some natural numbers $x, y$.
  – Base case: $n=4$. Show $P(4)$.
  – Inductive case: Assume $P(k)$ for arbitrary $k \geq 4$, show $P(k+1)$.
    • By I.H., $k = 2x + 5y$ for some $x, y$. Then, show there exist $x', y'$ such that $(k+1) = 2x' + 5y'$:
      • $k+1 = 2x + 5y + 1 = 2x + 5(y-1) + 6 = 2(x+3) + 5(y-1)$.
      • We’re done, right? Not if $k = 4$, or any number for which $y$ is 0. In other words, it doesn’t hold for all $k$.
    – Would strong induction help? Inductive case: Assume (as I.H.) $P(n)$ for all $4 \leq n < k$; show $P(k)$.
      • How would that proof go for $k = 5$?
Postage Stamps

• The difficulties on the previous slides suggest that we might need more coverage in our base case(s) to allow the inductive case to hold for all numbers.

• Claim: Any postage cost of 4 or more cents can be exactly covered by 2-cent and 5-cent stamps
  – Consider $P(n) = \text{“}n = 2x + 5y\text{” for some natural numbers } x, y\text{”}$
  – Base cases: $n=4$ and $n=5$. Show $P(4)$ and $P(5)$
  – Inductive case: Assume $P(k)$ and $P(k+1)$ for arbitrary $k \geq 4$, show $P(k+2)$.
    • How could we complete this proof?

Exercise: Might As Well Be Postage Stamps

• Claim: Every positive integer $n \geq 14$ can be written as a sum of 3s and 8s
  – i.e., there exist some $x, y$ s.t. $n = 3x + 8y$

• Proof: By induction!
  – (What’s the proof?)
Proving Statements about Recursively Defined Sets

• A related kind of induction is called structural induction, which can be used to prove claims about all items constructed by a recursive definition.

• To prove property P holds for all elements of a recursively defined set:
  – Base case(s): Show that P holds for every element in the basis for the recursive definition.
  – Inductive case(s): Show that every constructor in the definition preserves property P.

• Recall the definition of transitive closure:
  – \( R_0 = R \)
  – \( R_{n+1} = R_n \cup \{ (a,c) \mid \exists x \text{ s.t. } (a,x) \in R_n \text{ and } (x,c) \in R \} \)

• Claim: In the above definition, R is a subset of \( R_i \) for all i. Prove by structural induction.