Asymptotic Analysis-Ch. 3

Main idea: Running time is measured in the limit as the *input size* n grows to infinity.

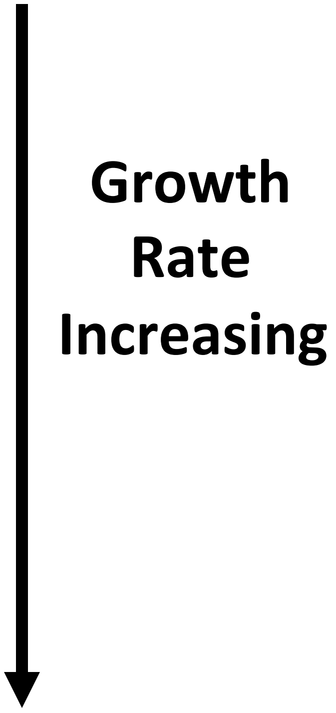
- Calculate algorithm running time in terms of its *rate of growth* with increasing problem size. To make this task easier, we can
  - **identify terms of highest order and ignore lower order terms**
  - **disregard multiplicative constants**

Saying an algorithm has running time $\theta(n^2)$ says that the *order of growth* of the running time is in the set of functions whose running time is n^2, a quadratic function of n

Asymptotic Analysis

- Names for classes of algorithms:

constant	$\theta(n^0) = \theta(1)$
logarithmic	$\theta(\lg n)$
polylogarithmic	$\theta(\lg^k n), k \geq 1$
linear	$\theta(n)$
linearithmic	$\theta(n\lg n)$
quadratic	$\theta(n^2)$
cubic	$\theta(n^3)$
polynomial	$\theta(n^k), k \geq 1$
exponential	$\theta(a^n), a > 1$



Asymptotic Analysis

Example: an algorithm with running time of order n^2 will "eventually" (i.e., for sufficiently large n) run slower than one with running time of order n, which in turn will eventually run slower than one with running time of order lgn.

Asymptotic analysis in terms of "Big Oh", "Theta", and "Big Omega" are the tools we will use to make these notions precise.

Note: Our conclusions will only be valid "in the limit" or "asymptotically". That is, they may not hold true for small values of n. And we really don't care about small values of n.

"Big Oh" - Upper Bounding Running Time

Definition: $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 1$ such that

$$f(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

Intuition:

- $f(n) \in O(g(n))$ means $f(n)$ is "of order at most", or "less than or equal to" $g(n)$ when we ignore small values of n and constants
- $f(n)$ is eventually trapped below (or = to) some constant multiple of $g(n)$
- some constant multiple of $g(n)$ is an <u>upper bound</u> for $f(n)$ (for large enough n)

"Big Oh" - Upper Bounding Running Time

Alternate Definition:

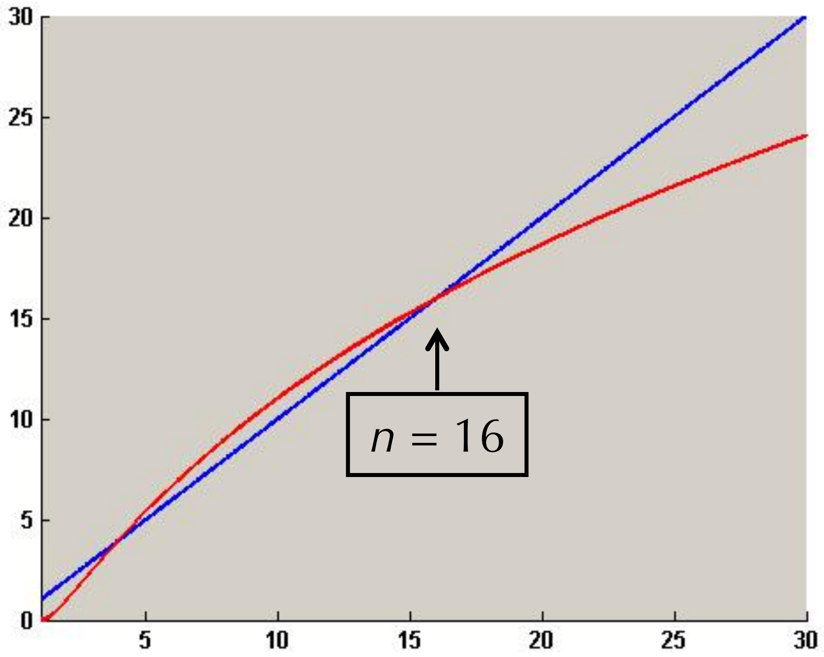
$O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 s.t.
$0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$

Intuition:

- $f(n) \in O(g(n))$ means f(n) is “of order at most”, or “less than or equal to” g(n) when we ignore small values of n and constants
- f(n) is eventually trapped below (or = to) some constant multiple of g(n)
- some constant multiple of g(n) is an upper bound for f(n) (for large enough n)

(note: n_0 must be at least 1)

Example: $(\lg n)^2$ is O(n)



$f(n) = (\lg n)^2$

$g(n) = n$

$(\lg n)^2 \leq n$ for all $n \geq 16$, so $(\lg n)^2$ is $O(n)$

Example: InsertionSort

INPUT:
An array A of *n* numbers $\{a_1, a_2, \ldots, a_n\}$

OUTPUT:
A permutation of input array $\{a_1', a_2', \ldots, a_n'\}$ such that $a_1' \leq a_2' \leq \ldots \leq a_n'$.

<u>InsertionSort(A)</u>

```
for j = 2 to A.length
    key = A[j]
    i = j - 1
    while i>0 and A[i]>key
      A[i+1] = A[i]
      i = i - 1
    A[i+1] = key
```

Time for execution on input array of length *n*:

- best-case: $b(n) \approx 5n - 4$
- worst-case: $w(n) \approx 3n^2/2 + 11n/2 - 4$

Insertion Sort - Time Complexity

Time complexities for insertion sort are:

- best-case: $b(n) = 5n - 4$
- worst-case: $w(n) = 3n^2/2 + 7n/2 - 4$

Questions:

1. is $b(n) = O(n)$? *Yes* $(5n - 4 < 6n)$ *for all* $n \geq 0$
2. is $w(n) = O(n)$? *No* $(3n^2/2 + 7n/2 - 4 \geq 3n)$ *for all* $n \geq 1$
3. is $w(n) = O(n^2)$? *Yes* $(3n^2/2 + 7n/2 - 4 \leq 4n^2)$ for all $n \geq 0$
4. is $w(n) = O(n^3)$? *Yes* $(3n^2/2 + 7n/2 - 4 \leq 2n^3)$ *for all* $n \geq 2$

Confused?

Basic idea: ignore constant factors and lower-order terms

- $617n^3 + 277x^2 + 720x + 7 \in \theta(?)$
- $200 \in \theta(?)$
- $(n\ (n+1))/2 \in \theta(?)$

Consider

$$f_1(n) = 5n^3 + 24n + 6$$

We claim that

$$f_1(n) = O(n^3)$$

Let $c = 6$ and $n_0 = 10$. Then

$$5n^3 + 24n + 6 \leq 6n^3$$

for every $n \geq 10$

If

$$f_1(n) = 5n^3 + 24n + 6$$

we have seen that

$$f_1(n) = O(n^3)$$

but $f_1(n)$ is not in $O(n^2)$, because no positive value for c or n_0 works.

"Big Omega" - Lower Bounding Running Time

Definition: $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 1$ such that

$$f(n) \geq cg(n) \quad \text{for all} \quad n \geq n_0.$$

Intuition:

- $f(n) \in \Omega(g(n))$ means $f(n)$ is "of order at least" or "greater than or equal to" $g(n)$ when we ignore small values of n .
- $f(n)$ is eventually trapped above (or = to) some constant multiple of $g(n)$
- some constant multiple of $g(n)$ is a lower bound for $f(n)$ (for large enough n)

"Big Omega" - Lower Bounding Running Time

Alternate Definition:

$\Omega(g(n))$ = { f(n) : there exist positive constants c and n_0 s.t.
$0 \le cg(n) \le f(n)$ for all $n \ge n_0$ }

Intuition:

- $f(n) \in \Omega(g(n))$ means f(n) is "of order at least" or "greater than or equal to" g(n) when we ignore small values of n .
- f(n) is eventually trapped above (or = to) some constant multiple of g(n)
- some constant multiple of g(n) is a lower bound for f(n) (for large enough n)

(note: n_0 must be at least 1)

Insertion Sort - Time Complexity

Time complexities for insertion sort are:

- o best-case: $b(n) = 5n - 4$
- o worst-case: $w(n) = 3n^2/2 + 7n/2 - 4$

Questions:

1. is $b(n) = \Omega(n)$? *Yes…* $(5n - 4 \geq 2n)$ *for all* $n_0 \geq 2$

2. is $w(n) = \Omega(n)$? *Yes…* $(3n^2/2 + 7n/2 - 4 \geq 3n)$ *for all* $n_0 \geq 1$

3. is $w(n) = \Omega(n^2)$? *Yes…* $(3n^2/2 + 7n/2 - 4 \geq n^2)$ *for all* $n_0 \geq 1$

4. is $w(n) = \Omega(n^3)$? *No …* $(3n^2/2 + 7n/2 - 4 < n^3)$ *for all* $n_0 \geq 3$

"Theta" - Tightly Bounding Running Time

$\theta(g(n))$ = { $f(n)$: there exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0$$ }

"Theta" - Tightly Bounding Running Time

Definition: $f(n) \in \theta(g(n))$ if there exist constants c_1, $c_2 > 0$ and $n_0 > 0$ such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for all } n \geq n_0.$$

Useful way to show "Theta" relationships:

❑ Show both a "Big Oh" and "Big Omega" relationship.

Insertion Sort - Time Complexity

Time complexities for insertion sort are:

- best-case: $b(n) = 5n - 4$
- worst-case: $w(n) = 3n^2/2 + 7n/2 - 4$

Questions:

1. is $b(n) = \theta(n)$? *Yes because* $b(n) = O(n)$ *and* $\Omega(n)$
2. is $w(n) = \theta(n)$? *No because* $w(n) \neq O(n)$
3. is $w(n) = \theta(n^2)$? *Yes because* $w(n) = O(n^2)$ *and* $\Omega(n^2)$
4. is $w(n) = \theta(n^3)$? *No because* $w(n) \neq \Omega(n^3)$

Asymptotic Analysis

- Classifying algorithms is generally done in terms of *worst-case* running time:

  - O (f(n)): Big Oh – asymptotic *upper* bound.
  - Ω (f(n)): Big Omega – asymptotic *lower* bound
  - θ (f(n)): Theta – asymptotic *tight* bound

"Little Oh" – Strict upper bound

Definition: $f(n) \in o(g(n))$ if for *every* $c > 0$, there exists some $n_0 \geq 1$ such that for all $n \geq n_0$, $\underline{f(n) < cg(n)}$.

Intuition:

- $f(n) \in o(g(n))$ means $f(n)$ is "strictly less than" any constant multiple of $g(n)$ when we ignore small values of n

- $f(n)$ is trapped below any constant multiple of $g(n)$ for large enough n

"Little Omega" – Strict Lower Bound

Definition: $f(n) \in \omega(g(n))$ if for every $c > 0$, there exists some $n_0 \geq 1$ such that for all $n \geq n_0$, $\underline{f(n) > cg(n)}$.

Intuition:

- $f(n) \in \omega(g(n))$ means $f(n)$ is "strictly greater than" any constant multiple of $g(n)$ when we ignore small values of n

- $f(n)$ is trapped above any constant multiple of $g(n)$ for large enough n

Using Limits to Determine Complexity

Showing "Little Oh" and "Little Omega" relationships:

$$f(n) \in o(g(n)) \quad \text{iff} \quad \lim_{n \to \infty} f(n) / g(n) = 0$$

$$f(n) \in \omega(g(n)) \quad \text{iff} \quad \lim_{n \to \infty} f(n) / g(n) = \infty$$

Showing Theta relationships

$$f(n) \in \theta(g(n)) \quad \text{iff} \quad \lim_{n \to \infty} f(n) / g(n) = c > 0$$

Analysis of PrefixAverages-v1

1. Create an array A such that length[A] = length[X] = n
2 . s = 0
3. **for (j** = 1 **to** length[X])
4. **s** = **s** + X[**j**]
5. A[**j**] = s / **j**
6. **return** A

$$\sum_{i=1}^{n} 1 = n - 1 + 1 = n$$

1. T(n) = θ(n)
2. Are there best- and worst-case inputs? No

Analysis of PrefixAverages-v2

```
Create an array A such that length[A] = n
for (j = 1 to n)
    a = 0
    for ( i = 1 to j)
        a = a + X[i]
    A[j] = a / j
return A
```

$$\sum_{j=1}^{n}\sum_{i=1}^{j}1=\sum_{j=1}^{n} j-1+1=\sum_{j=1}^{n} j=\frac{(n^{2}+n)}{2}$$

1. T(n) = $\theta(n^2)$
2. Are there best and worst case inputs? No

Basic asymptotic efficiency classes

Class	Name	Comments
1	Constant	Algorithm ignores input (i.e., can't even scan or print input)
lgn	Logarithmic	Cuts problem size by constant fraction on each iteration
n	Linear	Algorithm scans its input (at least); one or more non-nested loops
nlgn	Linearithmic	Some divide and conquer algorithms; best sorting time.
n^2	Quadratic	Loop inside loop = "nested loop"
n^3	Cubic	Loop inside nested loop
2^n	Exponential	Algorithm generates all subsets of n-element set
n!	Factorial	Algorithm generates all permutations of n-element set

Handy Asymptotic Facts

a) If $T(n)$ is a polynomial function of degree k, then $T(n) = O(n^k)$.

b) $\log^k n = (\log n)^k = O(n)$

c) $n^b = o(a^n)$ for any constants $a > 1$, $b > 0$.

d) $n! = o(n^n)$

e) $n! = \omega(2^n)$

f) $\lg(n!) = \Theta(n\lg n)$

g) $n^x = O(n^{x+\varepsilon})$, $a^n = O(a + \varepsilon)^n$

Oddball Running Time

- **Iterated logarithm function** ($\lg^* n$):
  - the number of times the log function must be iteratively applied before the result is less than or equal to 1
  - "log star of *n*"
  - *Very* slow growing, e.g. $\lg^*(2^{65536}) = 5$

  (and 2^{65536} is much larger than the number of atoms in the observable universe!!)

  eg: lg*2 = 1

  lg*4 = 2

  lg*16 = 3

  lg*65536 =4