Balanced Binary Search Trees

A binary search tree can implement any of the basic dynamic-set operations in O(h) time. These operations are O(lgn) if tree is "balanced".

BST balancing algorithms:

1st type: insert nodes as is done in the BST insert, then rebalance tree
- **Red-Black trees**: uses rotations & recoloring to balance tree
- **AVL trees**: uses rotations to balance tree

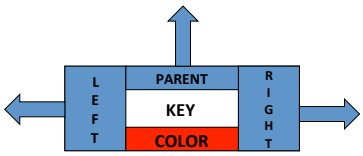
2nd type: allow more than one key per node of the search tree:
- **2-3 trees**: Uses <= 2 keys per node to keep tree balanced all the time (also 2-3-4 trees)
- **B-trees**: Lots of keys in each node. Good for storing large records of data

Red-Black Trees (RBT) (Ch. 13)

Red-Black tree: BST in which each node is colored red or black.

Constraints on the coloring and connection of nodes ensure that no root to leaf path is more than twice as long as any other, so tree is approximately balanced.

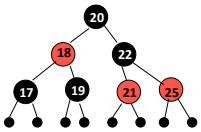
Each RBT node contains fields *left*, *right*, *parent*, *color*, and *key*.



Red-Black Properties

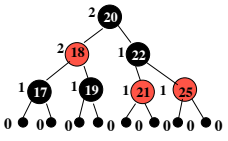
Red-Black tree properties:

1) Every node is either red or black.
2) The root is black.
3) Every leaf contains NIL and is black.
4) If a node is red, then both its children are black.
5) For each node x, all paths from x to its descendant leaves contain the same number of black nodes.



Black Height bh(x)

Black-height of a node x: bh(x) is the number of black nodes (including the NIL leaf) on the path from x to a leaf, *not counting x itself.*



Every node has a black-height, bh(x), labeled next to node.

For all NIL leaves, bh(x) = 0.

For root x, bh(x) = bh(T).

Red-Black Tree Height

Lemma 13.1: A red-black tree with n internal nodes has height at most 2lg(n+1).

Start with claim 1: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes.

Proof is by induction on the height of the node x.

Basis: height of x is 0 with bh(x) = 0. Then x is a leaf and its subtree contains $2^0-1=0$ internal nodes.

Inductive step: Consider a node x that has a positive height and 2 children. Each *child* of x has bh either equal to bh(x) (red child) or bh(x)-1 (black child).

Red-Black Tree Height

Claim 1: (cont) The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes.

We can apply the Inductive Hypothesis to the children of node x to find that the subtree rooted at each child of x has at least $2^{bh(x)-1} - 1$ internal nodes. Thus, the subtree rooted at x has at least $2(2^{bh(x)-1} - 1) + 1$ internal nodes = $2^{bh(x)} - 1$ internal nodes.

Red-Black Tree Height

Lemma13.1: (cont.)A red-black tree with n internal nodes has height at most 2lg(n+1).

Rest of proof of lemma: Let h be the height of the tree. By property 4 of RBTs, at least 1/2 the nodes on any root to leaf path are black. Therefore, the black-height of the root must be at least h/2.

Thus, by claim 1, $n \geq 2^{h/2} - 1$, so $n+1 \geq 2^{h/2}$ and, taking the log of both sides, $\lg(n+1) \geq h/2$, which means that $h \leq 2\lg(n+1)$.

Red-Black Tree Height

Since a red-black tree is a binary search tree. the dynamic-set operations for Search, Minimum, Maximum, Successor, and Predecessor for the binary search tree can be implemented as-is on red-black trees, and since they take O(h) time on a binary search tree, they take O(lgn) time on a red-black tree.

The operations Tree-Insert and Tree-Delete can also be done in O(lgn) time on red-black trees. However, after inserting or deleting, the nodes of the tree may have to be moved around to ensure that the red-black properties are maintained. The number of operations to move nodes around are constant at each level.

Operations on Red-Black Trees

All non-modifying bst operations (min, max, succ, pred, search) run in O(h) = O(lg*n*) time on red-black trees.

Insertion and deletion are more complex.

If we insert a node, what color do we make the new node?

* If red, the node might violate property 4.
* If black, the node might violate property 5.

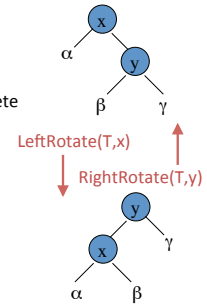
If we delete a node, what color was the node that was removed?

* Red? OK, since we won't have changed any black-heights, nor will we have created 2 red nodes in a row. Also, if node removed was red, it could not have been the root by prop. 2.
* Black? Could violate property 4, 5, or 2.

Red-Black Tree Rotations

Algorithms to restore RBT property to tree after Tree-Insert and Tree-Delete include right and left rotations and re-coloring nodes.

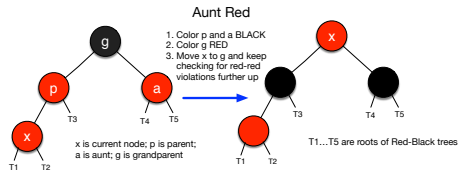
The number of rotations for insert and delete are constant, but they may take place at every level of the tree, so therefore the running time of insert and delete is O(lgn)



RBInsert(x): Parent and Aunt Red

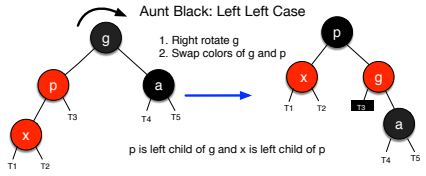
Perform standard binary search tree insertion and color x RED.

1. **If (x is the root) change x's color to BLACK and return (DONE)**
2. **If parent(x) is BLACK return (DONE)**
3. **If color of aunt(x) is RED:**
4. **Color parent(x) and aunt(x) BLACK**
5. **Color grandparent(x) RED**
6. **Set x to grandparent(x) and go to step 1**



RBInsert(x): Aunt Black

7. **else if color of aunt(x) is BLACK: there are 4 possible configurations for x, parent(x) and grandparent(x):**
   - i. **p is left child of g and x is left child of p (Left Left Case).**
   - ii. **p is left child of g and x is right child of p (Left Right Case).**
   - iii. **p is right child of g and x is right child of p (Right Right Case).**
   - iv. **p is right child of g and x is left child of p (Right Left Case).**



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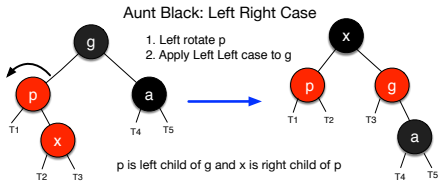

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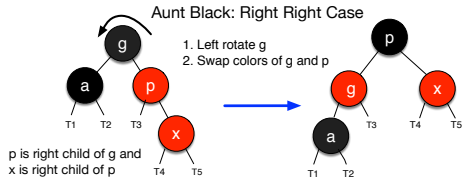

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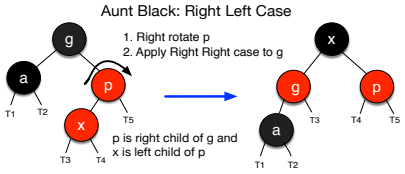


Operations on Red-Black Trees

The red-black tree procedures are intended to preserve the lg(n) running time of dynamic set operations by keeping the height of a binary search tree low, at most 2lgn.

Read sections 1 through 4 of Chapter 13.

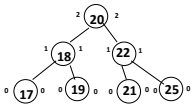
Given a binary search tree with red and black nodes, you should be able to decide whether it follows all the rules for a red-black tree (i.e., you should be able to determine whether a given binary search tree with node colors black and red is a red-black tree). You should also be able to give the black height of each node x in a red-black tree. I may give a programming assignment on RBTs after break.

AVL Trees

Developed by Russians Adelson-Velsky and Landis (hence AVL). This algorithm is not covered in our text but I've posted a reading on our web page.

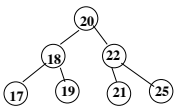
The AVL procedures keep the height of a binary search tree low. The balance factor of node x is the *difference in heights of nodes in x's left and right subtrees*

Definition: An AVL tree is a BST in which the difference in height between left and right subtrees is at most 1.



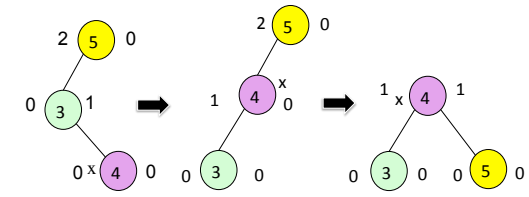
AVL Trees

Give the balance factor of all nodes in bst below:



Example of AVL rotations

Let x.p be the parent and x.p.p be the grandparent of x.

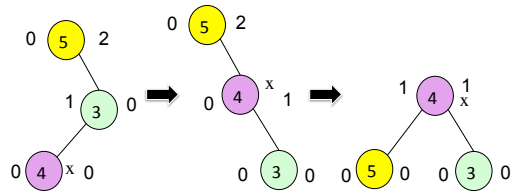


Case with 2-0 imbalance at x.p.p and 0-1 imbalance at x.p. (left rotation of x.p)

Case with 2-0 imbalance at x.p and 1-0 imbalance at x. (right rotation of x.p)

Tree Height-balanced at x

Example of AVL rotations



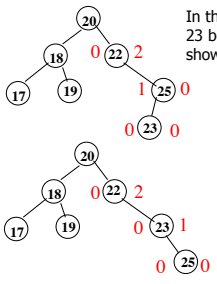
Case with 0-2 imbalance at x.p.p and 1-0 imbalance at x.p (right rotation of x.p)

Case with 0-2 imbalance at x.p and 0-1 imbalance at x (left rotation of x.p)

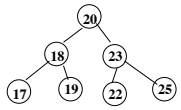
Tree Height-balanced at x

AVL Trees

In the case on the left below, the node 23 has just been inserted, creating an imbalance at the parent of its parent (node 22).



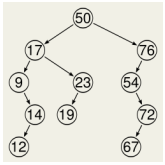
In this case, 25 is right-rotated to put 23 between 22 and 25, producing the tree shown below this tree.

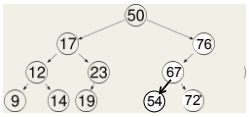


Then 23 is left-rotated to be the parent of left child 22 and right child 25. <*BALANCED*>

AVL trees

For each of these trees, indicate whether they are AVL trees by showing the height at each node.





Inserting nodes into AVL tree

Insert the following nodes into an AVL tree, in the order specified. Show the balance factor at each node as you add each one. When an imbalance occurs, specify the rotations needed to restore the AVL property. Nodes = <9, 5, 8, 3, 2, 4, 7>

AVL Tree Pros and Cons

1. Search is O(lgn) since AVL trees are always (nearly) balanced.

2. Insertion and deletion are also O(lgn).

The rotations may be needed along entire leaf to root path, whereas RB trees typically need fewer rotations.

2-3 Trees

Another set of procedures to keep the height of a binary search tree low.

Definition: A 2-3 tree is a tree that can have nodes of two kinds: **2-nodes** and **3-nodes**. A **2-node** contains a single key and has two children, exactly like any other binary search tree node. A **3-node** contains 2 values and has three children.

A 2-3 tree is always perfectly height balanced.

2-3 Tree Search

Search for a key k in a 2-3 tree:

1. Start at the root.
2. If the root is a 2-node (with only 1 key), look at the right node of the root if k is larger than key at root and to the left node of the root if k is smaller than key of root.
3. If the root is a 3-node (with 2 keys, K_1 and K_2), go to the left child if k is less than K_1 of root, to the middle child if k is greater than K_1 but less than K_2 of root, and go to the right child if k is greater than K_2 of root.

2-3 Tree Insert

Insert a key k in a 2-3 tree: (key always inserted in leaf node)

1. Start at the root.
2. Search for k until reaching a leaf.
    a) If leaf is a 2-node, insert k in proper position in leaf, either to the left or right of the key that already exists in the leaf, making it a 3-node.
    b) If leaf is a 3-node, temporarily make the leaf node have 3 keys (a 4-node): the smallest of the 3 keys is put in the left leaf, the largest key is put in the right leaf, and the middle key is promoted to the old leaf's parent. This may cause overload on the parent leaf and can lead to several node splits along the chain of the leaf's ancestors, possibly all the way to the root.

Inserting nodes into 2-3 tree

Insert the following nodes into a 2-3 tree, in the order specified. When an overload occurs, specify the changes needed to restore the 2-3 property. Nodes = <9, 5, 8, 3, 2, 4, 7>