Recursive Functions

13 October 2021
Where are we?
<table>
<thead>
<tr>
<th>number-grade</th>
<th>letter-grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>98</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>100</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>74</td>
<td>&quot;C&quot;</td>
</tr>
<tr>
<td>84</td>
<td>&quot;B&quot;</td>
</tr>
</tbody>
</table>

```
[list:
"A",
"A",
"C",
"B"]
```
A list is either:

- empty
- link(⟨item⟩, ⟨list⟩)
A list of one item, e.g.,

`[list: "A"]`,

is really a link between an item and the empty list:

`link("A", empty)`
[list:

"A", \rightarrow \text{link("A", }

"A", \rightarrow \text{link("A", }

"C", \rightarrow \text{link("C", }

"B"] \rightarrow \text{link("B", empty))))}
Is `link(3, 4)` a valid list?
We’ve seen convenient functions we can use to work with lists:

```python
>>> import lists as L

>>> lst = [list: "a", "b", "c"]

>>> L.map(lam(i): "item-" + i end, lst)
["item-a", "item-b", "item-c"]

>>> L.filter(lam(i): not(i == "a") end, lst)
["b", "c"]

>>> L.any(lam(i): i == "a" end, lst)
true

>>> L.all(lam(i): i == "a" end, lst)
false
```
But to write our own functions to process a list, item by item, we need to use the true form of a list and think *recursively*. 
Recursion is a technique that involves defining a solution or structure using itself as part of the definition.
fun my-sum(lst :: List<Number>) -> Number:
  ...

where:
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: ]) is 0
end
fun my-sum(lst :: List<Number>) -> Number:
  cases (List) lst:
      | empty => 0
      | link(f, r) => f + my-sum(r)
  end

where:
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: ]) is 0
end
fun my-sum(lst :: List<Number>) -> Number:

cases (List) lst:
  | empty => 0
  | link(f, r) => f + my-sum(r)
end

where:
my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
my-sum([list: 4]) is 4 + my-sum([list: ])
my-sum([list: ]) is 0
end

cases is a special form of conditional that we use to ask “which shape of data do I have?”
fun my-sum(lst :: List<Number>) -> Number:
    cases (List) lst:
        | empty => 0
        | link(f, r) => f + my-sum(r)
    end

where:

my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
my-sum([list: 4]) is 4 + my-sum([list: ])
my-sum([list: ]) is 0
end

Denotes the output of a function

Marks the expression to evaluate if the data has the shape on the left.
fun my-sum(lst :: List<Number>) -> Number:
    cases (List) lst:
        | empty => 0
        | link(f, r) => f + my-sum(r)
    end

where:
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: ]) is 0
When we call this function, it evaluates as:

\[
\text{my-sum(link(3, link(1, link(4, empty)))))} \\
3 + \text{my-sum(link(1, link(4, empty))))} \\
3 + 1 + \text{my-sum(link(4, empty))} \\
3 + 1 + 4 + \text{my-sum(empty)} \\
3 + 1 + 4 + 0
\]
Practice designing recursive functions
The function \texttt{any\textendash below\textendash 10} should return \texttt{true} if any member of the list is less than 10 and \texttt{false} otherwise.
We’ve already seen a higher-order function that lets us do this easily:

```haskell
fun any_below_10(lst :: List<Number>) -> Boolean:
    L.any(lam(x): x < 10 end, lst)
end
```
This is how you *should* write this function – higher-order functions like *any* are great!

We’ll implement it using recursion just for practice. After we’ve done that, we’ll be able to see how *any* is actually implemented!
fun any-below-10(lst :: List<Number>) -> Boolean:
    ...  
where:
    any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
    any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
    any-below-10([list: 4]) is (4 < 10)
    any-below-10([list: ]) is ...  
end

What goes here?
fun any-below-10(lst :: List<Number>) -> Boolean:
    ...

where:
    any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
    any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
    any-below-10([list: 4]) is (4 < 10)
    any-below-10([list: ]) is false
end
fun any-below-10(lst :: List<Number>) -> Boolean:
    ...
where:
    any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
    any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
    any-below-10([list: 4]) is (4 < 10)
    any-below-10([list: ]) is false
end
fun any-below-10(lst :: List<Number>) -> Boolean:
  ...

where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or any-below-10([list: 1, 4])
  any-below-10([list: 1, 4]) is (1 < 10) or any-below-10([list: 4])
  any-below-10([list: 4]) is (4 < 10) or any-below-10([list: ])
  any-below-10([list: ]) is false

end
fun any_below_10(lst :: List<Number>) -> Boolean:
  cases (List) lst:
    | empty => false
    | link(fst, rst) => (fst < 10) or any_below_10(rst)
  end

where:
  any_below_10([list: 3, 1, 4]) is (3 < 10) or any_below_10([list: 1, 4])
  any_below_10([list: 1, 4]) is (1 < 10) or any_below_10([list: 4])
  any_below_10([list: 4]) is (4 < 10) or any_below_10([list: ])
  any_below_10([list: ]) is false

end
Now that we’ve seen how to write `any-below-10`, we can use the same pattern to implement our own version of `any`. 
fun my-any(pred, lst :: List) -> Boolean:
  cases (List) lst:
    | empty => false
    | link(fst, rst) => pred(fst) or my-any(pred, rst)
  end
end
fun my-all(pred, lst :: List) -> Boolean:
  cases (List) lst:
    | empty => true
    | link(fst, rst) => pred(fst) and my-all(pred, rst)
  end
end
Thinking recursively
Any time a problem is structured such that the solution on larger inputs can be built from the solution on smaller inputs, recursion is appropriate.
All recursive functions have these two parts:

*Base case(s):*
What’s the simplest case to solve?

*Recursive case(s):*
What’s the relationship between the current case and the answer to a slightly smaller case?

You should be calling the function you’re defining here; this is referred to as a *recursive call.*
fun recursive-function(lst :: List) -> ...

cases (List) lst:
| empty => ...
| link(f, r) => ...
  ... recursive-function(r) ...
end
end
Each time you make a recursive call, you must make the input smaller somehow.

If your input is a list, you pass the rest of the list to the recursive call.
link("A",
link("A",
link("C",
link("B",
empty)))))
```python
>>> lst = [list: "item 1", "and", "so", "on"]
>>> lst.first
"item 1"
>>> lst.rest
[list: "and", "so", "on"]
```
cases (List) lst:
    | empty => ...
    | link(f, r) => ...
end

- First
- Rest
What happens if we *don’t* make the input smaller?
fun my-sum(lst :: List<Number>) -> Number:
    cases (List) lst:
        | empty => 0
        | link(f, r) => f + my-sum(r)
    end

where:
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: ]) is 0
end
fun my-sum(lst :: List<Number>) -> Number:
cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(lst)
end

where:
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: ]) is 0
end
When we call this function, it evaluates as:

\[
\text{my-sum(link(3, link(1, link(4, empty))))} \\
3 + \text{my-sum(link(3, link(1, link(4, empty))))} \\
3 + 3 + \text{my-sum(link(3, link(1, link(4, empty))))} \\
3 + 3 + 3 + \text{my-sum(link(3, link(1, link(4, empty))))} \\
3 + 3 + 3 + 3 + \text{my-sum(link(3, link(1, link(4, empty))))} \\
\ldots
\]

This isn’t going to end well.
When a recursive function never stops calling itself, it’s called *infinite recursion*.
Wrap-up practice
fun list-len(lst :: List) -> Number:
    doc: "Compute the length of a list"
    cases (List) lst:
        | empty => 0
        | link(f, r) => 1 + list-len(____)
    end
end
fun `list-len`(lst :: List) -> Number:
  doc: "Compute the length of a list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => 1 + list-len(r)
  end
end
fun list-product(lst :: List<Number>) -> Number:
  doc: "Compute the product of all the numbers in lst"
  cases (List) lst:
    | empty => 1
    | link(f, r) => ____ * list-product(r)
  end
end
fun list-product(lst :: List<Number>) -> Number:
    doc: "Compute the product of all the numbers in lst"
    cases (List) lst:
        | empty => 1
        | link(f, r) => f * list-product(r)
    end
end
fun is-member(lst :: List, item) -> Boolean:
  doc: "Return true if item is a member of lst"
  cases (List) lst:
    | empty => ______
    | link(f, r) =>
      (f == ______) or (is-member(______, ______))
  end
end
fun is-member(lst :: List, item) -> Boolean:
  doc: "Return true if item is a member of lst"
  cases (List) lst:
    | empty => false
    | link(f, r) =>
      (f == item) or (is-member(r, item)
  end
end
Final note

Lists, recursion, and cases syntax are not easy concepts to grasp separately, much less all together in a short time.

Don’t feel frustrated if it takes a little while for these to make sense. Give yourself time, be sure to practice working in Pyret, and ask questions.
Acknowledgments

This lecture incorporates material from:

Jonathan Gordon, Vassar College
Kathi Fisler, Brown University
Doug Woos, Brown University