cmpu 101 § 3 · Computer Science I

Lists and Recursion

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Dr. Marie desJardins

Dr. Marie desJardins on Fairness and Equity in Data Science: Challenges and Possibilities

Friday, Oct 7, 4:00 p.m.

Location:

New England 206

Dr. desJardins will talk about the current state of data science, machine learning, and AI; the future of these technologies; the importance of diversity for creating robust, effective engineering solutions; and how we can thoughtfully ensure that data science and computing will positively affect our lives and the lives of generations to come. Jointly sponsored by the Computer Science Department and Data Science & Society.

Reception to follow in Sanders Physics 105 from 5:00-6:00 p.m.

Where are we?

We’ve been working with tables for the past few weeks.

Last class we saw a new data type: lists.

```
>>> grades
```

number-grade	letter-grade
98	"A"
100	"A"
74	"C"
84	"B"

```
[list:
  "A",
  "A",
  "C",
  "B"]
```

```
>>> grades
```

number-grade	letter-grade
98	"A"
100	"A"
74	"C"
84	"B"

```
>>> grades.get-column("letter-grade")
[list:
  "A",
  "A",
  "C",
  "B"]
```

Columns in a table can contain a mix of different data types, e.g.,

```
table: grades
  row: 98
  row: 56
  row: 74
  row: "F"
  row: "A"
  row: "B"
end
```

And so can a list:

```
[list: 98, 56, 74, "F", "A", "B"]
```

However, we usually find it easier to work with a column where every value is of the same kind.

We can *annotate* the type of data in the column when we make a table:

```
table: col :: Number
  row: 1
  row: 2
  row: 3
end
```

```
table: col :: String
  row: "a"
  row: "b"
  row: "c"
end
```

Likewise, we'll most often have just one type of data in a list, and we can show that when we write the type annotation for a function:

For example,

```
[list: 1, 2, 3]
```

`List<Number>`

"a list of numbers"

```
[list: "a", "b", "c"]
```

`List<String>`

"a list of strings"

Much like the rows in a table, the items in a list have numeric indices:

```
                0    1    2
>>> lst = [list: "a", "b", "c"]
```

And we can access items using these indices:

```
>>> L.get(lst, 0)
"a"
>>> L.get(lst, 1)
"b"
```

```
                 0       1       2
>>> lst = [list: "a", "b", "c"]
```

The length of a list is always one more than the last item index:

```
>>> length(lst)
3
```

```
                  0        1        2
>>> lst = [list: "a", "b", "c"]
```

To check if an item is in a list, we can just ask if the list has it as a member:

```
>>> lst.member("c")
true
```

We used higher-order functions to work with tables, and we can do the same with lists:

Tables		*Lists*
`transform-column`	→	`map`

We used higher-order functions to work with tables, and we can do the same with lists:

Tables		*Lists*
`transform-column`	→	`map`
`filter-with`	→	`filter`

```
>>> lst = [list: "a", "b", "c"]
>>> filter(
      lam(i): not(i == "a") end,
      lst)
[list: "b", "c"]
```

```
>>> lst = [list: "a", "b", "c"]
>>> filter(
      lam(i): not(i == "a") end,
      lst)
[list: "b", "c"]
```

This is an anonymous (i.e., unnamed) function made using a lambda expression.

One difference to be aware of:

```
filter-with(⟨table⟩, ⟨function⟩)

filter(⟨function⟩, ⟨list⟩)
```

When you're working with a list, the function argument comes first.

At the end of last class, we considered what we could do if there wasn't a built-in function, so we needed to write a function that looked at each item in a list.

Designing list functions

How would we write a function that takes a list of numbers and returns its sum?

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is ...
end
```

We can have a string with no characters in it:

```
""
```

And, likewise, we can have a list with no items in it:

```
[list: ]
```

For these data types, these values are the equivalent of 0, the number representing no quantity.

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is 0
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4
  my-sum([list: 1, 4]) is 1 + 4
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4
  my-sum([list: 1, 4]) is 1 + 4
  my-sum([list: 3, 1, 4]) is 3 + 1 + 4
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list:         ]) is             0
  my-sum([list:        4]) is           4
  my-sum([list:     1, 4]) is       1 + 4
  my-sum([list:  3, 1, 4]) is   3 + 1 + 4
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list:        ]) is             0
  my-sum([list:       4]) is         4 + 0
  my-sum([list:    1, 4]) is     1 + 4 + 0
  my-sum([list: 3, 1, 4]) is 3 + 1 + 4 + 0
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list:        ]) is                 0
  my-sum([list:       4]) is             4 + my-sum([list: ])
  my-sum([list:    1, 4]) is         1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

The secret nature of lists

Writing our input as `[list: 3, 1, 4]` is a lie.

It’s just a shorthand for the real structure of a list.

In its secret heart, Pyret knows there are only two ways of making a list.

A list is either:

```
empty
```

```
link(⟨item⟩, ⟨list⟩)
```

A list of one item, e.g.,

```
[list: "A"],
```

is really a link between an item and the empty list:

```
link("A", empty)
```

```
[list:

  "A",  ──────────────▶  link("A",

  "A",  ──────────────▶    link("A",

  "C",  ──────────────▶      link("C",

  "B"]  ──────────────▶        link("B",

                                 empty))))
```

Is **`link(3, 4)`** a valid list?

Is `link(3, 4)` a valid list? 

Designing functions using the definition of a list

To write our own functions to process a list, item by item, we need to use the true form of a list and think *recursively*.

Recursion is a technique that involves defining a solution or structure using itself as part of the definition.

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

`cases` *is like a special* **`if`** *statement that we use to ask "which* ***shape*** *of data do I have?"*

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

*If the list is **`empty`**, do one thing.*

*If it's a **`link`**, do another thing.*

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

Denotes the output of a function

Marks the expression to evaluate if the data has the shape on the left.

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

This gives names for referring to the arguments to **`my-sum`**.

And this is giving names for referring to the arguments to **`link`**.

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      0

    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      0

    | link(f, r) =>
      f + my-sum(r)

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      0

    | link(f, r) =>
      f + my-sum(r)

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(r)
  end
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

```
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(r)
  end
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

When we call this function, it evaluates as:

```
  my-sum(link(3, link(1, link(4, empty))))
→ 3 + my-sum(link(1, link(4, empty)))
→ 3 + 1 + my-sum(link(4, empty))
→ 3 + 1 + 4 + my-sum(empty)
→ 3 + 1 + 4 + 0
```

Thinking recursively

Any time a problem is structured such that the solution on larger inputs can be built from the solution on smaller inputs, recursion is appropriate.

All recursive functions have these two parts:

Base case(s):

What's the simplest case to solve?

Recursive case(s):

What's the relationship between the current case and the answer to a slightly smaller case?

You should be calling the function you're defining here; this is referred to as a *recursive call*.

```
fun recursive-function(lst :: List) -> ...:
  cases (List) lst:
    | empty =>
      ...
    | link(f, r) =>
      ... recursive-function(r) ...
  end
end
```

Base case

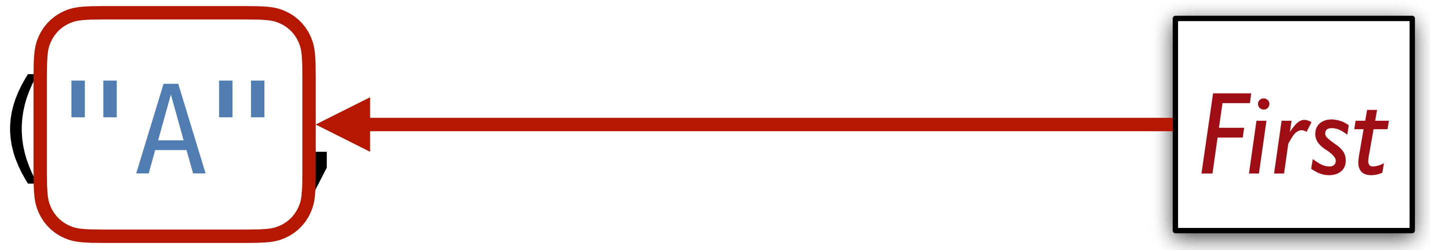
Recursive case

Each time you make a recursive call, you must make the input smaller somehow.

If your input is a list, you pass the ***rest*** of the list to the recursive call.

```
link("A",
  link("A",
    link("C",
      link("B",
        empty))))
```

```
link("A",
  link("A",
    link("C",
      link("B",
        empty))))
```



```
link("A",
```

First

```
link("A",
    link("C",
        link("B",
            empty))))
```

Rest

```
>>> lst = [list: "item 1", "and", "so", "on"]
>>> lst.first
"item 1"
>>> lst.rest
[list: "and", "so", "on"]
```

```
cases (List) lst:
  | empty => ...
  | link(f, r) => ...
end
```

First *Rest*

What happens if we *don't* make the input smaller?

```
fun my-sum(lst :: List<Number>) -> Number:
  cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(r)
  end
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

Recursive call on the rest of the input list

```
fun my-sum(lst :: List<Number>) -> Number:
  cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(lst)
  end
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
```

Recursive call on the original input list

When we call this function, it evaluates as:

```
  my-sum(link(3, link(1, link(4, empty))))
→ 3 + my-sum(link(3, link(1, link(4, empty))))
→ 3 + 3 + my-sum(link(3, link(1, link(4, empty))))
→ 3 + 3 + 3 + my-sum(link(3, link(1, link(4, empty))))
...
```

This isn't going to end well.

When a recursive function never stops calling itself,
it's called *infinite recursion*.

Practice designing recursive functions

The function **`any-below-10`** should return `true` if any member of the list is less than 10 and `false` otherwise.

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"

  ...


where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
  any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
  any-below-10([list: 4]) is (4 < 10)
  any-below-10([list: ]) is ...
end
```

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"

  ...

where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
  any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
  any-below-10([list: 4]) is (4 < 10)
  any-below-10([list: ]) is ...
end
```

What goes here?

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"

  ...


where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
  any-below-10([list: 1, 4]) is (1 < 10) or (4 < 10)
  any-below-10([list: 4]) is (4 < 10)
  any-below-10([list: ]) is false
end
```

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"

  ...


where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or (1 < 10) or (4 < 10)
  any-below-10([list:    1, 4]) is             (1 < 10) or (4 < 10)
  any-below-10([list:       4]) is                         (4 < 10)
  any-below-10([list:        ]) is false
end
```

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"

  ...

where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or any-below-10([list: 1, 4])
  any-below-10([list:    1, 4]) is (1 < 10) or any-below-10([list: 4])
  any-below-10([list:       4]) is (4 < 10) or any-below-10([list: ])
  any-below-10([list:        ]) is false
end
```

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less than 10"
  cases (List) lst:
    | empty => false
    | link(f, r) => (f < 10) or any-below-10(r)
  end
where:
  any-below-10([list: 3, 1, 4]) is (3 < 10) or any-below-10([list: 1, 4])
  any-below-10([list:    1, 4]) is (1 < 10) or any-below-10([list: 4])
  any-below-10([list:       4]) is (4 < 10) or any-below-10([list: ])
  any-below-10([list:        ]) is false
end
```

Now that we've seen how to write **`any-below-10`**, we can use the same pattern to implement a higher-order function where we can ask if any item in a list satisfies *some predicate*.

```
fun my-any(fn :: Function, lst :: List) -> Boolean:
  doc: "Return true if the function fn is true for any
item in the given list."
  cases (List) lst:
    | empty => false
    | link(f, r) => fn(f) or my-any(fn, r)
  end
end
```

```
fun my-any(fn :: Function, lst :: List) -> Boolean:
  doc: "Return true if the function fn is true for any
item in the given list."
  cases (List) lst:
    | empty => false
    | link(f, r) => fn(f) or my-any(fn, r)
  end
end


fun my-all(fn :: Function, lst :: List) -> Boolean:
  doc: "Return true if the function fn is true for
every item in the given list."
  cases (List) lst:
    | empty => true
    | link(f, r) => fn(f) and my-all(fn, rst)
  end
end
```

```
fun any-below-10(lst :: List<Number>) -> Boolean:
  doc: "Return true if any number in the list is less
than 10"
  any(lam(x): x < 10 end, lst)
where:
  any-below-10([list: 3, 1, 4]) is true
  any-below-10([list: 11, 14]) is false
  any-below-10([list: ]) is false
end
```

This is how you *should* write this function – use built-in higher-order functions like **any** when you can!

Wrap-up practice

```
fun list-len(lst :: List) -> Number:
  doc: "Compute the length of a list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => 1 + list-len(____)
  end
end
```

```
fun list-len(lst :: List) -> Number:
  doc: "Compute the length of a list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => 1 + list-len(r)
  end
end
```

```
fun list-product(lst :: List<Number>) -> Number:
  doc: "Compute the product of all the numbers in lst"
  cases (List) lst:
    | empty => 1
    | link(f, r) => ____ * list-product(r)
  end
end
```

```
fun list-product(lst :: List<Number>) -> Number:
  doc: "Compute the product of all the numbers in lst"
  cases (List) lst:
    | empty => 1
    | link(f, r) => f * list-product(r)
  end
end
```

```
fun is-member(item, lst :: List) -> Boolean:
  doc: "Return true if item is a member of lst"
  cases (List) lst:
    | empty => ______
    | link(f, r) =>
      (f == ______) or (is-member(______, ______)
  end
end
```

```
fun is-member(item, lst :: List) -> Boolean:
  doc: "Return true if item is a member of lst"
  cases (List) lst:
    | empty => false
    | link(f, r) =>
      (f == item) or (is-member(item, r)
  end
end
```

Final note

Lists, recursion, and **cases** syntax are not easy concepts to grasp separately, much less all together in a short time.

Don't feel frustrated if it takes a little while for these to make sense. Give yourself time, be sure to practice working in Pyret, and ask questions.

Class code:

`https://tinyurl.com/101-2022-10-05`

Acknowledgments

This lecture incorporates material from:

Kathi Fisler, Brown University

Doug Woos, Brown University