Where are we?
How was the lab?

We’ve been working with tables for the past few weeks.

Last class we saw a new data type: lists.
<table>
<thead>
<tr>
<th>number-grade</th>
<th>letter-grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>98</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>100</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>74</td>
<td>&quot;C&quot;</td>
</tr>
<tr>
<td>84</td>
<td>&quot;B&quot;</td>
</tr>
</tbody>
</table>

```python
>>> grades

["A", "A", "C", "B"]
```
grades

<table>
<thead>
<tr>
<th>number-grade</th>
<th>letter-grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>98</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>100</td>
<td>&quot;A&quot;</td>
</tr>
<tr>
<td>74</td>
<td>&quot;C&quot;</td>
</tr>
<tr>
<td>84</td>
<td>&quot;B&quot;</td>
</tr>
</tbody>
</table>

grades.get-column("letter-grade")

["A", "A", "C", "B"]
We used higher-order functions to work with tables, and we can do the same with lists:

\[ \text{Tables} \quad \xrightarrow{\text{transform-column}} \quad \text{Lists} \quad \xrightarrow{\text{map}} \]
We used higher-order functions to work with tables, and we can do the same with lists:
```python
>>> lst = [list: "a", "b", "c"]
>>> filter(
    lam(i): not(i == "a") end,
    lst)
["b", "c"]
```

This is an anonymous (i.e., unnamed) function made using a lambda expression.
Numbers, strings, images, Booleans, tables, and lists let us represent many kinds of real data quite naturally.

But there are times when we’re going to want something a bit different.
Defining structured data
Imagine that we’re doing a study on communication patterns among students.

We don’t have access to the messages the students sent – hopefully they’re encrypted! – but we have *metadata* for each message:

- sender
- recipient
- day of the week
- time (hour and minute)
This kind of metadata might sound uninteresting, but it can tell us a lot!

Recommended reading:

John Bohannon, “Your call and text records are far more revealing than you think”, Science, 2016
Imagine that we’re doing a study on communication patterns among students.

We don’t have access to the messages the students sent – maybe they’re encrypted! – but we have metadata for each message:

- sender
- recipient
- day of the week
- time (hour and minute)

How should we store this data?
We could have a table, e.g.,

<table>
<thead>
<tr>
<th>sender :: String</th>
<th>recipient :: String</th>
<th>day :: String</th>
<th>time :: …</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>…</td>
</tr>
</tbody>
</table>
We could have a table, e.g.,

<table>
<thead>
<tr>
<th>sender :: String</th>
<th>recipient :: String</th>
<th>day :: String</th>
<th>time :: String</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>&quot;4:55&quot;</td>
</tr>
</tbody>
</table>
We could have a table, e.g.,

<table>
<thead>
<tr>
<th>sender</th>
<th>recipient</th>
<th>day</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>295</td>
</tr>
</tbody>
</table>
We could have a table, e.g.,

<table>
<thead>
<tr>
<th>sender :: String</th>
<th>recipient :: String</th>
<th>day :: String</th>
<th>time :: List</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>[list: 4, 55]</td>
</tr>
</tbody>
</table>
We could have a table, e.g.,

<table>
<thead>
<tr>
<th>sender :: String</th>
<th>recipient :: String</th>
<th>day :: String</th>
<th>hour :: Number</th>
<th>minute :: Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>4</td>
<td>55</td>
</tr>
</tbody>
</table>
If we use multiple columns, we can access the components independently, by name, but if we use a single column, all of the “time” data is in one place.
To resolve this trade-off, we add structure: We can have a single data type that has named parts.
data Time:
    | time(hours :: Number, mins :: Number)
end
The name of the data type

data Time:
  | time(hours :: Number, mins :: Number)
end
data Time:
  time(hours :: Number, mins :: Number)
end

A constructor function that builds the data type
data Time:
| time(hours :: Number, mins :: Number)
end

The components of the data
After defining the data type,

```java
data Time:
    | time(hours :: Number, mins :: Number)
end
```

we can call `time` to build `Time` values,

```python
>>> noon = time(12, 0)
>>> half-past-three = time(3, 30)
```

and we can use dot notation to access the components:

```python
>>> noon.hours
12
>>> half-past.mins
30
```
Our table could now be:

<table>
<thead>
<tr>
<th>sender :: String</th>
<th>recipient :: String</th>
<th>day :: String</th>
<th>time :: Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;4015551234&quot;</td>
<td>&quot;8025551234&quot;</td>
<td>&quot;Mon&quot;</td>
<td>time(4, 55)</td>
</tr>
</tbody>
</table>
Conditional data
data **Time:**
  | time(hours :: Number, mins :: Number)
end

*The only way to make Time is to call the time() constructor function.*
But we can also define *conditional data*, where there are multiple varieties of the data.
The varieties can just be fixed values, e.g.,

data Day:
    | sunday
    | monday
    | tuesday
    | wednesday
    | thursday
    | friday
    | saturday
end
Or they can be separate constructors, e.g.,

```haskell
data Message:
  | direct(sender :: String,
  |  recipient :: String,
  |  message :: String)
  | group(sender :: String,
  |  recipients :: List<String>,
  |  message :: String)
end
```
Or we can mix these together, e.g.,

```plaintext
data Name:
  | name(first :: String, last :: String)
  | anonymous
end
```
Recursive data definitions
Last week we worked with *lists* – ordered sequences of items, equivalent to a column in a table.
Much like the rows in a table, the items in a list have numeric indices:

```
>>> lst = ['a', 'b', 'c']
```

And we can access items using these indices:

```
>>> lst[0]
'a'

>>> lst[1]
'b'
```

Much like the rows in a table, the items in a list have numeric indices:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
\end{array}
\]

```python
>>> lst = ["a", "b", "c"]
```

And we can access items using these indices:

```python
>>> lst[0]
"a"
>>> lst[1]
"b"
```
But writing the list as [list: "a", "b", "c"] is just a convenient deception!
In its secret heart, Pyret knows there are only two ways of making a list.

A list is either:

- empty or
- linking an item to another list.
That is, a list is a kind of conditional data:

```plaintext
data List:
  | empty
  | link(first :: Any, rest :: List)
end
```
So, a list of one item, e.g.,

```
list: "A"
```

is really a link between an item and the empty list:

```
link("A", empty)
```
list:

"A",

"A",

"A",

"C",

"B"]

link("A",

link("A",

link("C",

link("B",

empty))))
Recursion
1+1=2
2+1=3
3+1=4
4+1=5

Count all the buses

Count all the buses

Count one bus
Recursion is a programming technique where a problem is solved by solving a smaller version of the same problem, unless that smaller version is simple enough to solve directly.
We call the small version that can be solved directly the \textit{base case} of the recursive problem.
To write our own functions to process a list, item by item, we need to use the true form of a list and think recursively.
Designing functions using the definition of a list
How would we write a function that takes a list of numbers and returns its sum?
fun my-sum(lst :: List<Number>) -> Number:
   doc: "Return the sum of the numbers in the list"
   ...
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

where:
  my-sum([list: ]) is ...
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
  where:
  my-sum([list: ]) is 0
end
fun **my-sum**(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4
  my-sum([list: 1, 4]) is 1 + 4

end
fun my-sum(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4
    my-sum([list: 1, 4]) is 1 + 4
    my-sum([list: 3, 1, 4]) is 3 + 1 + 4
end
fun my-sum(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4
    my-sum([list: 1, 4]) is 1 + 4
    my-sum([list: 3, 1, 4]) is 3 + 1 + 4
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
...
where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + 0
  my-sum([list: 1, 4]) is 1 + 4 + 0
  my-sum([list: 3, 1, 4]) is 3 + 1 + 4 + 0
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"
  ...
where:
  my-sum([list:  ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list:  ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"
    ...

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

cases (List) lst:
  | empty =>
  |     ...
  | link(f, r) =>
  |     ...

end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun **my-sum**(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"

cases (List) lst:
    | empty =>
    ...

    | link(f, r) =>
    ...

d | my-sum([list: ]) is 0
  d | my-sum([list: 4]) is 4 + my-sum([list: ])
  d | my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  d | my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
    ...
    | link(f, r) =>
    ...

  end

  where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
  end

If the list is empty, do one thing.

If it's a link, do another thing.
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty => ...
    | link(f, r) => ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end

Marks the expression to evaluate if the data has the shape on the left.

Denotes the output of a function.
fun my-sum(lst :: List<Number>) -> Number:

doc: "Return the sum of the numbers in the list"

cases (List) lst:
    | empty =>
        ...
    ...
    | link(f, r) =>
        ...
end

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end

This gives names for referring to the arguments to my-sum.

And this is giving names for referring to the arguments to link.
fun my-sum(lst : List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

  cases (List) lst:
    | empty =>
      ...
    |
    | link(f, r) =>
      ...

  end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst : List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"

    cases (List) lst:
        | empty =>
        | 0
        |
        | link(f, r) =>
        | ...
        |
    end

    where:
        my-sum([list: []]) is 0
        my-sum([list: 4]) is 4 + my-sum([list: []])
        my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
        my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
    end
fun my-sum(lst :: List<Number>) -> Number:
  doc: "Return the sum of the numbers in the list"

cases (List) lst:
  | empty =>
  | empty =>
  | link(f, r) =>
      f + my-sum(r)
end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"

    cases (List) lst:
        | empty =>
        | 0

        | link(f, r) =>
        | f + my-sum(r)

end

where:
    my-sum([list: ])) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:

doc: "Return the sum of the numbers in the list"

cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(r)
end

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
fun my-sum(lst :: List<Number>) -> Number:
    doc: "Return the sum of the numbers in the list"
    cases (List) lst:
        | empty => 0
        | link(f, r) => f + my-sum(r)
    end

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end
When we call this function, it evaluates as:

\[
\text{my-\text{sum}}(\text{link}(3, \text{link}(1, \text{link}(4, \text{empty}))))
\]

\[
\rightarrow 3 + \text{my-\text{sum}}(\text{link}(1, \text{link}(4, \text{empty})))
\]

\[
\rightarrow 3 + 1 + \text{my-\text{sum}}(\text{link}(4, \text{empty}))
\]

\[
\rightarrow 3 + 1 + 4 + \text{my-\text{sum}}(\text{empty})
\]

\[
\rightarrow 3 + 1 + 4 + 0
\]
Thinking recursively
Any time a problem is structured such that the solution on larger inputs can be built from the solution on smaller inputs, recursion is appropriate.
All recursive functions have these two parts:

*Base case(s):*

What’s the simplest case to solve?

*Recursive case(s):*

What’s the relationship between the current case and the answer to a slightly smaller case?

You should be calling the function you’re defining here; this is referred to as a *recursive call.*
fun recursive-function(lst :: List) -> ...:
cases (List) lst:
  | empty =>
  ...
  | link(f, r) =>
    ... recursive-function(r) ...
end
end
Each time you make a recursive call, you must make the input smaller somehow.

If your input is a list, you pass the rest of the list to the recursive call.
>>> lst = [list: "item 1", "and", "so", "on"]
>>> lst.first
"item 1"
>>> lst.rest
[list: "and", "so", "on"]
cases (List) lst:
  | empty  => ...
  | link(f, r) => ...
end

First
Rest
What happens if we *don’t* make the input smaller?
fun my-sum(lst :: List<Number>) -> Number:
  cases (List) lst:
  | empty => 0
  | link(f, r) => f + my-sum(r)
end

where:
  my-sum([list: ]) is 0
  my-sum([list: 4]) is 4 + my-sum([list: ])
  my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
  my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end

Recursive call on the rest of the input list.
fun my-sum(lst :: List<Number>) -> Number:
    cases (List) lst:
    | empty => 0
    | link(f, r) => f + my-sum(lst)
end

where:
    my-sum([list: ]) is 0
    my-sum([list: 4]) is 4 + my-sum([list: ])
    my-sum([list: 1, 4]) is 1 + my-sum([list: 4])
    my-sum([list: 3, 1, 4]) is 3 + my-sum([list: 1, 4])
end

Recursive call on the original input list.
When we call this function, it evaluates as:

\[
\text{my-sum(link(3, link(1, link(4, empty))))}
\rightarrow 3 + \text{my-sum(link(3, link(1, link(4, empty))))}
\rightarrow 3 + 3 + \text{my-sum(link(3, link(1, link(4, empty))))}
\rightarrow 3 + 3 + 3 + \text{my-sum(link(3, link(1, link(4, empty))))}
\]

... 

\textit{This isn't going to end well.}
When a recursive function never stops calling itself, it’s called *infinite recursion*. 
Wrap-up practice
fun list-len(lst :: List) -> Number:
    doc: "Compute the length of a list"
    cases (List) lst:
        | empty => 0
        | link(f, r) => 1 + list-len(____)
    end
end
fun list-len(lst :: List) -> Number:
  doc: "Compute the length of a list"
  cases (List) lst:
    | empty => 0
    | link(f, r) => 1 + list-len(r)
  end
end
fun list-product(lst :: List<Number>) -> Number:
  doc: "Compute the product of all the numbers in lst"
  cases (List) lst:
    | empty => 1
    | link(f, r) => ____ * list-product(r)
  end
end
fun **list-product** (lst :: List<Number>) -> Number:

doc: "Compute the product of all the numbers in lst"

cases (List) lst:
  | empty => 1
  | link(f, r) => f * list-product(r)

end
end
fun is-member(item, lst :: List) -> Boolean:
   doc: "Return true if item is a member of lst"
   cases (List) lst:
      | empty => ______
      | link(f, r) =>
         (f == ______) or (is-member(______, ______))
   end
end
fun is-member(item, lst :: List) -> Boolean:
  doc: "Return true if item is a member of lst"
  cases (List) lst:
    | empty => false
    | link(f, r) =>
      (f == item) or (is-member(item, r)
  end
end
Final note

Lists, recursion, and cases syntax are not easy concepts to grasp separately, much less all together in a short time.

Don’t feel frustrated if it takes a little while for these to make sense. Give yourself time, be sure to practice working in Pyret, and ask questions.
Class code:
tinyurl.com/101-2023-02-13
Acknowledgments

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