CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – W F 1:30pm
   Lab – F 3:30pm

Lecture Meeting Location: SP 105
   Lab Meeting Location: SP 309

Business

• HW2-Lookahead out
   – Full HW2 expected out today (or tomorrow), due Feb. 22

• Please read Ch.2.1, 2.2.2, 2.3.1, 2.4, 2.5, and 2.7.1 in your textbook
   – You can skip section 2.3.2, but people interested in databases might want to read it anyway

(note: graph images in this lecture are from the textbook “Artificial Intelligence: A Modern Approach” by Stuart Russell and Peter Norvig)
Exercise

• Claim: For any sets A and B, \((A \cap B) \subseteq (A \cup B)\).
  – Prove it, or give a counterexample.

Ordered Pairs and n-tuples

• By design / definition, sets are good for asking questions of membership, but not for questions of relative ordering of elements
• A different structure, an n-tuple, represents elements and their ordering

• Definition: An ordered pair is a pair of elements expressed in parenthesis—e.g., \((0,0)\), \((9,17)\), (Jon Stewart, Stephen Colbert)
  – Order matters, so \((17, 9)\) is not the same as \((9, 17)\)
  – Similarly, order matters, so \((0,0)\) does not contain redundant elements—the 0s are distinct from each other, by position
  – How could we state the criterion for identity for ordered pairs?
• This generalizes to n-tuples with more than 2 elements
  – E.g., \((0,0),(3,4,5)\) are both 3-tuples; \((72, 86, 94, 86, 76, 66, 72)\) is a 7-tuple
Set Product
(Cartesian Product)

- Given sets A and B, ordered pairs can represent elements from those sets and which set each element came from
- Definition: The set product (or Cartesian product) of sets A and B is
  \[ A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \} \]
  - That is, it’s the set of all pairs s.t. the first element is in A and the second element is in B
  - Note: This generalizes to more than two sets. \( A_1 \times A_2 \times \cdots \times A_n \) is all n-tuples s.t. the first element is from \( A_1 \), the second from \( A_2 \), …, and the n’th from \( A_n \)
- Exercises
  - What’s \( \{0, 1\} \times \{1, 2, 3\} \)?
  - What’s \( \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \)?
  - Let \( A = \{2\} \times \{1, \ldots, 28\} \), \( B = \{4,6,9,11\} \times \{1, \ldots, 30\} \), and \( C = \{1,3,5,7,8,10,12\} \times \{1, \ldots, 31\} \).
    What is \( A \cup B \cup C \)?

“Cogito ergo product”? Are you sure you don’t mean Cartesian sum?
Yes, I’m sure.

Binary Relations

- A relation, intuitively enough, expresses the relation between elements of various sets. More formally…
- For sets A and B, a binary relation from A to B (or over A x B) is a subset of A x B. (That's the Cartesian product of A and B)
  - i.e., it is a set of ordered pairs of the form (a,b) where a ∈ A and b ∈ B
  - … thus, it relates elements of A to elements of B
  - Note: If a relation R is a subset of A x A, we say it is a relation over A
- More generally, for sets \( A_1, \ldots, A_n \), an n-place relation over \( A_1, \ldots, A_n \) is a subset of the set product \( A_1 \times \cdots \times A_n \)
Directed Graphs

- One way of representing a binary relation is a directed graph, which indicates ordered relations between elements.

- A directed graph $G$ is a pair $(V,E)$ where $V$ is a finite set of vertices (singular: vertex), and $E$ is a binary relation on $V \times V$.
  - $E$ is called the edges (or edge set) of graph $G$.

- Exercise: For graph (a), what are the sets $V$ and $E$?

Graphs, and Inverse of a Relation

- Given a relation $R$, the inverse $R^{-1}$ of $R$ is the set of all ordered pairs $(b,a)$ s.t. $(a,b) \in R$.
  - You can think of the inverse as reversing the direction of $R$.
  - Your textbook calls this the converse of $R$; I’m more familiar with calling it the inverse.

- Questions:
  - Are $R^{-1}$ and $R$ disjoint?
  - What’s the inverse of the relation in (c), below? (express it as ordered pairs, not a graph)
## Properties of Relations

- Some useful properties of relations! Definitions:
  - A relation $R$ over a set $A$ is **reflexive** if for all $a \in A$, $(a, a) \in R$
    - What would a reflexive relation be over the set $\{1, 2, 3\}$?
    - (If it’s clear in context what the set $A$ is, we might simply say that the relation $R$ is reflexive, in that context)
  - A relation $R$ is **symmetric** if whenever $(a, b) \in R$, $(b, a) \in R$
  - A relation $R$ is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

- Exercise: Consider the following relations over $\{1, 2, 3, 4\}$
  - $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
  - $R_2 = \{(1,1), (1,2), (2,1)\}$
  - $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
  - $R_4 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
  - Which of these are reflexive? symmetric? transitive?

## More Vocabulary and Properties of Relations

- For relation $R$:
  - **Domain** $\text{dom}(R) = \{a \mid \exists b \text{ s.t. } (a, b) \in R\}$—that is, the domain is all elements that are the first element of a pair in $R$
  - **Range** $\text{range}(R) = \{b \mid \exists a \text{ s.t. } (a, b) \in R\}$—that is, the range is all elements that are the second element of a pair in $R$

- Recall that relations are sets of n-tuples; binary relations are sets of ordered pairs. So, applying ideas from subsets:
  - A relation $R$ could be a **subrelation** of a relation $S$
  - The **empty relation** has no elements—it is the same as $\emptyset$

- Exercises:
  - What is $\text{dom}(A \times B)$? What is $\text{range}(A \times B)$?
  - What does it mean if relations $R$ and $S$ are **disjoint**? If $A = \{1, 2, 3\}$, what is an example of disjoint relations over $A$?
  - Consider a relation $R$ and its inverse $R^{-1}$. How do $\text{dom}(R)$ and $\text{range}(R)$ relate to $\text{dom}(R^{-1})$ and $\text{range}(R^{-1})$?
Digression: Undirected Graphs

- There are also undirected graphs
- An undirected graph is a pair $G=(V,E)$ where
  - $V$ (vertices) is a finite set, and
  - $E$ (edges) is a set of unordered pairs on $V \times V$

- What are $V$ and $E$ for undirected graph (b), above?