CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T R 3:10pm
Lab – M 3:10pm

Lecture Meeting Location: OH 162
Lab Meeting Location: SP 309

Business

• Options for making up missed classes
• Change lecture rooms for first half of the semester?
• HW1 due Tuesday, February 13
  – (i.e., at the beginning of class that day—as always, please see the HW sheet for information about HW deadlines)
  – In proofs, use formal definitions / notation where appropriate for clarity and rigorous expression of the relevant ideas
• See links from course webpage Additional Notes / Readings for two documents:
  – A document about induction (containing the proof from the last lecture!)
  – A document about the equational proof style we use for inductions
• Reading: Please finish Ch. 1 from your textbook (Ch. 2 is next!)
  – Your book talks about sets and diagrams in Ch.1.2.3; it’s interesting, but I won’t cover it in class
Before We Hit Empty: A Logical Progression

- Basic ideas from propositional logic are showing up in “logic boxes” in your reading

- Before we go further into sets, an overview of these basics of boolean expressions and propositional reasoning

Propositions

- Defn: proposition – a statement that has the property of truth or falsity

- Propositions are the key elements to represent, analyze, or explain declarative knowledge

Propositions:
- Washington, D.C. is the capital of the USA.
- Poughkeepsie is the capital of New York.
- \(1 + 1 = 2\)
- \(2 + 2 = 3\)

Non-Propositions:
- What time is it?
- Pass the salt.
- \(x + 1 = 2\)
- \(x^2 y + 5^2 z\) presuming values for \(x, y, z\) are not given / known

The first and third of these are true; the second and fourth are false.
Propositional operators

- Recall: *proposition* – a statement that has the property of truth or falsity
  - Often, we use *propositional letters* (or *variables*) to represent propositions: e.g., \( p \) stands for “Poughkeepsie is the capital of NY”

- There are several *operators* (sometimes called *boolean operators*) that can construct new propositions from old ones
  - *Negation* (“not”): if \( P \) is a proposition, \( \neg P \) is a proposition
  - *Conjunction* (“and”): \( P \) and \( Q \)
  - *Disjunction* (“or”): \( P \) or \( Q \)
  - *Implication* (“if – then”): if \( P \) then \( Q \)
  - *Equivalence* (“is equal / equivalent to”): \( P \) iff \( Q \)
    - Equivalence can also be written as “if and only if”

Propositional operator: Negation

- Whatever the value of \( p \), True or False, the value of proposition \( \neg p \) (written \( \neg p \)) is the opposite
  - If \( p \) is “Today is Monday,” \( \neg p \) is “It is not the case that today is Monday,” or more simply “Today is not Monday.”
- Negation can be expressed with a *truth table*

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Propositional operator: Conjunction

• Conjunction—the “and” operator
  – Whatever the values of propositions $p$, $q$, conjunction $p$ and $q$ (written $p \land q$ or $p \& \& q$) is also a proposition
  – If $p$ is “Today is Monday” and $q$ is “It is snowing today,” then $p \land q$ is “Today is Monday and it is snowing today.”
    • $p \land q$ is true on snowy Mondays and false on any day that is not Monday, and on any day that is Monday but not snowing

• Conjunction values as a truth table

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Propositional operator: Disjunction

• Disjunction—the “or” operator
  – Whatever the values of propositions $p$, $q$, disjunction $p$ or $q$ (written $p \lor q$ or $p \mid\mid q$) is also a proposition
  – If $p$ is “Today is Monday” and $q$ is “It is snowing today,” then $p \lor q$ is “Today is Monday or it is snowing today.”
    • $p \lor q$ is true on any day that is a Monday or on which it is snowing – including snowy Mondays (it is not exclusive) – and false only on days that are not Mondays on which it is not snowing

• Disjunction values as a truth table

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Propositional operator: Implication

- Implication—the “if…then” operator (also called conditional)
  - Whatever the values of propositions p, q, implication if p then q
    (written $p \rightarrow q$) is also a proposition
  - If p is “Today is Monday” and q is “It is snowing today,” then $p \rightarrow q$ is
    “If today is Monday then it is snowing today.”
  - Vocabulary: in $p \rightarrow q$, p is called the hypothesis (or antecedent) and q is
called the conclusion (or consequent)
- Implication values as a truth table

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Really? These are the truth values for implication?
They look like the values for $(\neg p \lor q)$! (Exercise: Check for yourselves!!)

Sounds if-y: Material Implication

- Meaning for implication symbol $\rightarrow$ in propositional
  logic is referred to as material implication
  - It says that $p \rightarrow q$ is False exactly when p is True and q is False
  - Not the same as every meaning of “if…then” in English, but it’s what’s used
  in logic

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Examples of material implication and natural language usage:
- Politician says: “If I am elected, then I will fix the environment”
  - False if the speaker is elected and doesn’t fix the environment
  - True if, e.g., the speaker doesn’t get elected
- “If today is Friday, then 2 + 2 = 4”
  - True no matter what day it is
  - True except on Fridays, even though $2 + 2 = 5$ is false!
Properties of operators

- Logical operators have an *order of operations* just like mathematical operators
  - From high to low: negation; conjunction; disjunction; implication
    - Conjunction is kinda like multiplication; disjunction is kinda like addition
    - Math: \(-k \times (x + y)\)
    - Logic: \(\neg p \land (q \lor r)\)
  - Also similarly, disjunction and conjunction are *commutative* and *associative*
    - Associative: e.g., \(p \land q \land r\) is \((p \land q) \land r\) is \(p \land (q \land r)\)
    - Commutative: e.g., \(p \land q\) is \(q \land p\)
      - similar with disjunction
  - Implication is *right-associative*
    - \(p \Rightarrow q \Rightarrow r\) is \(p \Rightarrow (q \Rightarrow r)\)

The *biconditional* (or *equivalence*) operator

- The *biconditional* (or *equivalence*) operator:
  - If \(p\) and \(q\) are propositions, then \(p \iff q\) is a proposition, read as “\(p\) if and only if \(q\)”
  - \(p \iff q\) is true exactly when \(p\) and \(q\) have the same truth values
- What does the truth table for \(\iff\) look like?
- How could we define the biconditional in terms of operators we already know (not, and, or, if… then)?

The equivalence operator can also be written as \(\equiv\) or \(==\) in other contexts.
Exercise: Evaluating boolean expressions

- Defn: Propositions are boolean-valued expressions—i.e., their values are either True or False
- Boolean expressions are evaluated like any other mathematical expressions

Examples: Let \( p = \text{True} \), \( q = \text{False} \), \( r = \text{True} \). What do the following expressions evaluate to?
1. \( p \land \neg r \)
2. \( q \lor \text{False} \)
3. \( p \rightarrow q \)
4. \( q \leftrightarrow p \)
5. \( q \leftrightarrow \neg \text{True} \)
6. \( r \lor (p \land q) \)
7. \( (p \lor r) \rightarrow ((p \lor q) \land r) \)
8. \( \text{True} \rightarrow r \)

Compound propositions and their truth tables

- Just as we use truth tables to understand meanings of propositional operators, we can also use them to understand compound propositions
  - The truth table for \( (p \lor \neg q) \rightarrow (p \land q) \):

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Exercise: Truth tables for compound propositions

• What are truth tables for the following propositions?
  1. \( p \rightarrow \neg p \)
  2. \( p \leftrightarrow \neg p \)
  3. \( (p \rightarrow q) \land (\neg p \rightarrow q) \)
  4. \( (p \lor q) \land r \)
  5. \( p \rightarrow (\neg q \lor r) \)

Material Implication and Proofs of “If-then” Propositions

• Material implication review: \( P \rightarrow Q \) is equivalent to \( \neg P \lor Q \)
  – When \( P \) is false, \( P \rightarrow Q \) is true, regardless of \( Q \)
  – When \( P \) is true, \( P \rightarrow Q \) is true exactly when \( Q \) is true
• Now, imagine we want to prove statement \( P \rightarrow Q \) is always true
  – We know it’s true whenever \( P \) is false—no need to prove more!
  – All that’s left is to prove that whenever \( P \) is true, \( Q \) is also true
• Proof strategy for an implication (an “if-then” proposition):
  – Assume the antecedent (\( P \))
  – Prove the consequent (\( Q \))

This structure is a direct proof of the implication. For CS145, please consider this as your primary proof strategy when proving implications.
Contrapositive and Proof of “If-then” Propositions

- Material implication review: $P \implies Q$ is equivalent to $\neg P \vee Q$
- Therefore, $\neg Q \implies \neg P$ is equivalent to $P \implies Q$.

- Note that $\neg (\neg Q) = Q$—i.e., double negation leaves the value unchanged. Then:
  
  $\neg Q \implies \neg P = \neg (\neg Q) \vee \neg P = Q \vee \neg P = \neg P \vee Q = P \implies Q$

- Thus, whenever we're proving implication $P \implies Q$, we could equivalently prove $\neg Q \implies \neg P$.

- Vocabulary: The expression $\neg Q \implies \neg P$ is called the contrapositive of $P \implies Q$; the proof technique is proof by contraposition (or proof by contrapositive).

Example: Comparing Direct Proof and Proof by Contraposition

- Consider numbers $x$ and $y$.
- Claim: If both of $x$, $y > 2$, then $x + y > 4$
- Direct proof:
  - Assume: $x > 2$ and $y > 2$
  - Show: $x + y > 4$
- Contrapositive proof:
  - Assume: NOT $x + y > 4$—i.e., Assume: $x + y \leq 4$
  - Show: NOT both of $x$, $y$ is greater than 2—i.e.,
    Show: at least one of $x, y$ is less than or equal to 2

Which of these proof strategies seems more intuitive to you?
Example: Comparing Direct Proof and Proof by Contraposition—The Flip Side

- Consider numbers x and y.
- Claim: If $x + y < 4$, then at least one of $x$, $y$ is less than 2
- Direct proof:
  - Assume: $x + y < 4$
  - Show: at least one of $x$, $y$ is less than 2
- Contrapositive proof:
  - Assume: NOT at least one of $x$, $y$ is less than 2—i.e.,
    Assume: both of $x,y$ are greater than or equal to 2
  - Show: NOT $x + y < 4$—i.e., Show: $x + y \geq 4$

Which of these proof strategies seems more intuitive to you?

Take-home Message About Contrapositive and Implication

- For CS145: Please be able to prove implications directly
  - i.e., when proving $P \Rightarrow Q$, assume $P$ is true, and under that assumption, prove $Q$ is true
    It is possible that some CS145 exercises might explicitly ask for a direct (i.e., not contrapositive) proof. Please be comfortable with this proof technique!
  - This is the most straightforward way of approaching such a proof (see your textbook, pg. 30)...
  - … but it is not always the clearest. In some cases, proof using contraposition can make for cleaner proofs.
- How can you tell when to use direct proof vs. contraposition?
  - To some extent, that’s a matter of proof style—the more proofs you do, the more intuitive this will become!

Please feel free to ask your Prof. any questions about this or other matters of proof style!
End Of Logical Progression:
Back To Sets

• Those fundamental logical / boolean operators are part of the way we reason about many things, including sets

• (There’s an example coming up soon)