CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – W F 1:30pm
Lab – F 3:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW2 out, due Feb. 22
• HW1 back today

• Please read Ch.2.1, 2.2.2, 2.3.1, 2.4, 2.5, and 2.7.1 in your textbook
  – You can skip section 2.3.2, but people interested in databases might want to read it anyway
Other Properties of Relations

• Recall definitions:
  – A relation $R$ over a set $A$ is **reflexive** if for all $a \in A$, $(a,a) \in R$
  – A relation $R$ is **symmetric** if whenever $(a,b) \in R$, $(b,a) \in R$
  – A relation $R$ is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

• Relations can also be described as, in some sense (be very careful!), having the opposites of those properties (note the $\notin$ symbols!):
  – A relation $R$ over a set $A$ is **irreflexive** if for all $a \in A$, $(a,a) \notin R$
  – A relation $R$ is **asymmetric** if whenever $(a,b) \in R$, $(b,a) \notin R$
  – A relation $R$ is **intransitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \notin R$

• Note that, e.g., not being reflexive is not the same as being irreflexive!

• Exercises:
  – Give a relation over $\{1,2,3\}$ that is neither reflexive nor irreflexive
  – Give a relation over $\{1,2,3\}$ that is neither symmetric nor asymmetric
  – Give a relation over $\{1,2,3\}$ that is neither transitive nor intransitive

Equivalence Relations

• Recall definitions:
  – A relation $R$ over a set $A$ is **reflexive** if for all $a \in A$, $(a,a) \in R$
  – A relation $R$ is **symmetric** if whenever $(a,b) \in R$, $(b,a) \in R$
  – A relation $R$ is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

• When a relation $R$ has all of these properties, it is called an **equivalence relation**

• If $R$ is an equivalence relation, then it induces **equivalence classes** on the elements
  – For equivalence relation $R$ and any element $a$, let $C_a$ stand for all elements related to element $a$ in $R$—that is, $C_a = \{ b | (a,b) \in R \}$
  – Then, $C_a = C_b$ exactly when $(a,b) \in R$!

• What are some equivalence relations over the natural numbers?
  – Verify all three properties. What are the equivalence classes?
  – What do the graphs of these equivalence relations look like?
Partitions

• Intuitively, what does it mean to you if something—or a collection of things—is partitioned?

• How could we write that definition in rigorous, formal notation?

Computer scientists can often do this kind of thing—studying what is meant by something and coming up with a rigorous, formal definition that fits its applications. That way, even a computer can work with that definition!

Partitions

• Intuitively, a partition of a set $S$ is a way of breaking $S$ into a collection of non-empty subsets $S_1, S_2, \ldots, S_n$ such that
  – the subsets include all of $S$;
  – … and the subsets don’t overlap.

• Below, a partition of the set $S = \{0, 1, \ldots, 10\}$

• What’s the connection between a partition and an equivalence relation?
Partitions

More formally, a definition of a *partition*:

- Let $A$ be a non-empty set, and let $\{B_i\}_{i \in I}$ be an indexed collection of non-empty subsets of $A$ ($I$ is called an index set)

- ... Then, $\{B_i\}_{i \in I}$ is a *partition* of $A$ iff
  1. $\{B_i\}_{i \in I}$ is a pairwise-disjoint collection (do you remember this definition?)
  2. $\{B_i\}_{i \in I}$ exhausts $A$ (see definition below)
     - Definition: We say that $\{B_i\}$ *exhausts* $A$ iff $(\bigcup_{i \in I} B_i) = A$
       - That is, $\forall a \in A$, $\exists i \in I$ s.t. $a \in B_i$

How does this compare to our intuitive sense(s) of what it means for something to be partitioned?

Equivalence Relations and Partitions

- Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  - Every equivalence relation over $A$ determines a partition over $A$
  - Every partition over $A$ determines an equivalence relation over $A$
  - Thus, in some sense, they're doing the same thing!

- Claim: Every equivalence relation over $A$ determines a partition over $A$
  - Proof:

- Claim: Every partition over $A$ determines an equivalence relation over $A$
  - Proof: