CS 145 – Foundations of Computer Science

Professor Eric Aaron

**Lecture** – T R 3:10pm

**Lab** – M 3:10pm

**Lecture Meeting Location**: OH 162

**Lab Meeting Location**: SP 309

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**Business**

- HW2 in already
  - (HW2 solutions out before HW2 is graded?)
- HW3 out today
  - We may not have covered everything on it (i.e., for exercise 3) by the end of today’s class; in that case, we’ll get to the rest on Thursday
  - HW3 due March 7 / March 8 (see assignment sheet)

*Note: Sometimes when I say a HW is due on a certain day, that may mean that the programming part is due the day before—e.g., saying HW3 is due March 8*

- Please read Ch.2.1, 2.2.2, 2.3.1, 2.4, 2.5, and 2.7.1 in your textbook
  - You can skip section 2.3.2, but people interested in databases might want to read it anyway
- Make-up lab session (as emailed)—attendance is OPTIONAL
  - Wed., Feb. 28, 4:30-5:45pm
Partitions

More formally, a definition of a \emph{partition}:

- Let $A$ be a non-empty set, and let $\{B_i\}_{i \in I}$ be an indexed collection of non-empty subsets of $A$ \footnote{\textit{I is called an index set}}

- ... Then, $\{B_i\}_{i \in I}$ is a \emph{partition} of $A$ iff
  1. $\{B_i\}_{i \in I}$ is a \emph{pairwise-disjoint} collection \footnote{do you remember this definition?}
  2. $\{B_i\}_{i \in I}$ \emph{exhausts} $A$ \footnote{see definition below}

  - Definition: We say that $\{B_i\}$ \emph{exhausts} $A$ iff $(\bigcup\{B_i\}_{i \in I}) = A$
    - That is, $\forall a \in A, \exists i \in I$ s.t. $a \in B_i$

  How does this compare to our intuitive sense(s) of what it means for something to be partitioned?

Equivalence Relations and Partitions

- Equivalence relations and partitions can be viewed as different ways of expressing the same thing:
  - Every equivalence relation over $A$ determines a partition over $A$
  - Every partition over $A$ determines an equivalence relation over $A$
  - Thus, in some sense, they're doing the same thing!

- Claim: Every equivalence relation over $A$ determines a partition over $A$
  - Proof?:

- Claim: Every partition over $A$ determines an equivalence relation over $A$
  - Proof?:
Equivalence Relations and Partitions

• Claim: Every partition over A determines an equivalence relation over A
  – Proof: Let \( \{B_i\}_{i \in I} \) be a partition of A. Thus, each \( B_i \) is non-empty, and the collection of sets \( \{B_i\}_{i \in I} \) is pairwise disjoint and exhausts A.
  – Define relation \( E = \{(a,b) \mid \exists i \text{ s.t. } a, b \text{ are both in } B_i \} \)
    • We say \( E \) is the relation over A associated with (or induced by) the partition \( \{B_i\}_{i \in I} \).
  – Claim: \( E \) is an equivalence relation over A. Must show \( E \) is...
    1. Reflexive: (Proof?)
    2. Symmetric: (Proof?)
    3. Transitive: (Proof?)

Equivalence Relations and Partitions

• Claim: Every equivalence relation over A determines a partition over A
  – Proof: Let \( E \) be an equivalence relation over A.
    Consider the sets \( E_a \) s.t. \( E_a = \{x \mid (a,x) \in E\} \) — i.e., the equivalence classes of \( E \).
  – Then, it remains to prove that the collection of all sets \( \{E_a\} \) are a partition of A. To fit the definition of partition, we must show
    1. Each \( E_a \) is non-empty
    2. The collection \( \{E_a\} \) is pairwise-disjoint
    3. Together, the \( E_a \) exhaust \( A \) (i.e., every element in \( A \) is in at least one \( E_a \))
(Flighty, but Not A Digression)

• Imagine that a directed graph $G = (V,E)$ represented a flight map, with edges indicating roads (or flights)
  – I.e., for $v, w \in V$, $(v,w) \in E$ iff there’s a direct flight from $v$ to $w$

• So, the graph directly answers the “direct flight” question
  – What other questions might you want to ask?

You might want to ask “What kind of an airline has two flights into Salt Lake City but only one flight into Los Angeles?”
But perhaps there are other questions to ask too?
**Where Can I Go From Here…?**

- Imagine that a directed graph represents a flight map, with edges indicating flights (and direction)
  - Edge relation represents, for each vertex v in V, what other vertices w were reachable by non-stop flight from v
  - I.e., for v, w ∈ V, (v,w) ∈ E iff there’s a direct flight from v to w
- So, the graph directly answers the “direct flight” question
  - What would answer the “where can I go from here, in *any* number of stops?” question?
  - That is, what would contain all (v,x) if there are *any number* of flights that could eventually get someone from v to x, through *any number* of stops?

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**Transitive Closure**

- The *transitive closure* $R^*$ of a (binary) relation $R$ is the *unique smallest* relation that is transitive and that includes $R$
- One way of looking at it:
  - Define $R^*$ by: $(x_1,x_n)$ is in $R^*$ iff there is a “path” $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_n$ from $x_1$ to $x_n$ in $R$—i.e., where every “step” $(x_i, x_j)$ on that path is in $R$
  - This is essentially the *finite path* definition of $R^*$ on pg. 52 of your textbook
- Another way of looking at it:
  - *Top-down definition:* $R^* = \bigcap \{ S : R \subseteq S \subseteq A^2 \text{ and } S \text{ is transitive} \}$—i.e.,
    - $R^*$ is the intersection of all possible transitive relations (over A, in this case) that contain $R$
More Transitive Closure

- The transitive closure \( R^* \) of a (binary) relation \( R \) is the unique smallest relation that is transitive and that includes \( R \).

- One more way of looking at it, the “bottom-up” definition:
  Given a relation \( R \), consider the following recursive definition of relations \( R_i \):
  - \( R_0 = R \)
  - \( R_{n+1} = R_n \cup \{(a,c) \mid \exists x \text{ s.t. } (a,x) \in R_n \text{ and } (x,c) \in R\} \)
  - (What is the relationship between the indices of the \( R_i \) and the finite path definition on the previous slide?)

- … then, we define \( R^* \) to be the union of all the \( R_i \) sets:
  - \( R^* = \bigcup \{ R_n : n \text{ is a natural number} \} \)

- Intuition: Keep adding pairs whenever \((a,x)\) is in the partially constructed \( R^* \) relation and \((x,c)\) is in \( R \)

I recommend this definition—it is very useful!