CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T R 3:10pm
Lab – M 3:10pm

Lecture Meeting Location: OH 162
Lab Meeting Location: SP 309
Business

• HW4 due Apr. 4 / Apr. 5 (as always, see assignment sheet for exact deadlines)

• Reading: Ch.4.1-4.6
  – Our coverage of the material will be different from that in the textbook, but it’s good to see the textbook’s presentation, as well

• Reading: Prof. Hunsberger’s document “The Natural Numbers, Induction, and Numeric Recursion”
  – Posted on the Additional Notes / Readings page of the CS145 website
Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…
  – What would the base case(s) be?
  – What would the inductive case(s) be? **Note: We proved that every natural number is either 0 or S(m) for some m. How can that help us in this recursive definition?**
More Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…

- Recall: We proved that every natural number is either 0 or S(m) for some m.

• Definition of addition:
  1. Case z=0 -- For all n in N, n + 0 = n
  2. Case z=S(m) -- For all n in N, z = S(m) for some m in N:
     \[ n + S(m) = S(n + m) \]

• Let’s prove something with that definition!
  – Claim: This addition function is \textit{associative} (i.e., \( a + (b + c) = (a + b) + c \), for all \( a, b, c \) in \( N \))
  – Proof: ??
Proof: Associativity of Addition

- Definition of addition:
  1. Case $z=0$ -- For all $n \in \mathbb{N}$, $n + 0 = n$
  2. Case $z=S(m)$ -- For all $n \in \mathbb{N}$, $z = S(m)$ for some $m \in \mathbb{N}$: $n + S(m) = S(n + m)$

- Let’s prove something with that definition—associativity!
  - To prove: For all $a, b, c \in \mathbb{N}$, $a + (b + c) = (a + b) + c$
  - Proof: Prove by induction on $c$.
    - Let $P(c)$ be the proposition: For all $a, b \in \mathbb{N}$, $a + (b + c) = (a + b) + c$
    - Base—$c=0$.
    - $P(0)$ is the proposition: For all $a, b \in \mathbb{N}$, $a + (b + 0) = (a + b) + 0$
    - To prove: $P(0)$—for arb’ly chosen $a, b$, show $a + (b + 0) = (a + b) + 0$
      - Start with the left hand side: $a + (b + 0) = a + b$, because $(b + 0) = b$ (by equation 1 of the definition of addition, with $b$ in place of $n$); then, $a + b = (a + b) + 0$, also by equation 1, with $(a + b)$ in place of $n$. Thus, $a + (b + 0) = (a + b) + 0$, completing the proof of the base case.
Proof: Associativity of Addition

- Definition of addition:
  1. Case z=0 -- For all n in N, n + 0 = n
  2. Case z=S(m) -- For all n in N, z = S(m) for some m in N: n + S(m) = S(n + m)
- Let’s prove something with that definition—associativity!
  - Inductive case—c=S(y) for some y in N. Then, for that y, assume P(y) and prove P(S(y)): For all a, b in N, a + (b + S(y)) = (a + b) + S(y).
    - Consider arbitrarily chosen a, b. Show a + (b + S(y)) = (a + b) + S(y).
    - Start with the left hand side: a + (b + S(y)) = a + S(b + y), by eqn 2 of the definition of addition, with b in place of n and y in place of m;
    - Then, a + S(b + y) = S(a + (b + y)), by eqn 2, with a in place of n and (b + y) in place of m;
    - Then, S(a + (b + y)) = S((a + b) + y), because a + (b + y) = (a + b) + y, by the inductive hypothesis; (Be sure to explicitly note where the I.H. is used!)
    - Then, S((a + b) + y) = (a + b) + S(y), by eqn 2 with (a + b) in place of n and y in place of m;
    - Thus, a + (b + S(y)) = (a + b) + S(y), completing the inductive case.
Proof: Associativity of Addition

- Definition of addition:
  1. Case \( z = 0 \) -- For all \( n \) in \( \mathbb{N} \), \( n + 0 = n \)
  2. Case \( z = S(m) \) -- For all \( n \) in \( \mathbb{N} \), \( z = S(m) \) for some \( m \) in \( \mathbb{N} \):
     \[ n + S(m) = S(n + m) \]

- Let’s prove something with that definition—associativity!
  - To prove: For all \( a, b, c \) in \( \mathbb{N} \), \( a + (b + c) = (a + b) + c \)
  - Proof: Prove by induction on \( c \)
    - Let \( P(c) \) be the proposition: For all \( a, b \) in \( \mathbb{N} \), \( a + (b + c) = (a + b) + c \)
    - Base—\( c = 0 \). Prove \( P(0) \).
    - Inductive case—\( c = S(y) \) for some \( y \). Prove \( P(S(y)) \).
    - Both cases are proved, on the previous slides. Therefore, the claim holds
      for all numbers \( c \) in \( \mathbb{N} \), by induction. (Specifically, by the Axiom of
      induction!)
Theorem: George

- Definition of addition:
  1. Case $z=0$: For all $n$ in $\mathbb{N}$, $n + 0 = n$
  2. Case $z=S(m)$: For all $n$ in $\mathbb{N}$, $z = S(m)$ for some $m$ in $\mathbb{N}$: $n + S(m) = S(n + m)$

- Theorem: George—$x + 0 = 0 + x$ for all natural numbers $x$

- Proof: Let $P(n)$ be the proposition $n + 0 = 0 + n$
  - To prove: $P(n)$ holds for all numbers $n$ in $\mathbb{N}$. Proof by induction on $n$.

- Base case: $P(0)$. To prove …

- Inductive case: Assume for (arbitrarily chosen) $k$ that $P(k)$ holds. Then, prove $P(S(k))$. To prove…

What is the Inductive Hypothesis, in this case?