CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – T R 3:10pm
Lab – M 3:10pm

Lecture Meeting Location: OH 162
Lab Meeting Location: SP 309

Business

• HW7-Lookahead out
  – Full programming assignment out already
  – The remainder will be out tonight!
  – (As always, see HW sheet for deadlines)

• Reading: Ch. 6
  – We may not cover all the material in chapter 6, but it’s worth reading anyway

• Lecture notes from yesterday’s make-up lecture are now posted on the Additional Notes / Readings course webpage
Two Definitions of Independence, The Sequel—Still Independent

- Let $p$ be a probability function on a (finite) sample space $S$, and let $A, B$ be events.
  - Then, we say $A$ is independent of $B$ iff $p(A) = p(A \mid B)$
  - ... or $p(B) = 0$
- Alternatively, we say $A$ is independent of $B$ iff
  $$p(A \cap B) = p(A) \cdot p(B)$$
- Are these definitions equivalent?
- A note about complements:
  1. $A$ and $B$ are independent iff...
  2. $A$ and $\neg B$ are independent iff...
  3. $\neg A$ and $B$ are independent iff...
  4. $\neg A$ and $\neg B$ are independent

Hint: Yes they are equivalent. I’ll leave it to you to go through the proof as an exercise... please come to me if you have questions on it!

That is, the conditions numbered 1-4 here are all equivalent. See Ch. 6.6 in your book for more info!

An Example

- RC Online sells refurbished computers. 5% of the computers they sell are defective.
- A customer just bought 3 computers from them.
- Assuming the 3 computers are a random sample, and their performance—including any defects—is independent of each other, what are:
  - The probability that all 3 computers are defective?
  - The probability that none of the computers is defective?
  - The probability that at least one of the computers is defective?
Bayes’ Theorem

- We’ve seen that these two conditional probabilities…
  - $p(A \mid B)$, the probability of event $A$ given $B$
  - $p(B \mid A)$, the probability of event $B$ given $A$
  .. are not equal to each other. They are, however, related!
- Bayes’ Theorem: Assuming $p(A)$ and $p(B)$ both non-zero:
  $$p(A \mid B) = \frac{p(B \mid A) \cdot p(A)}{p(B)}$$
  - That is, if we know the prior (non-conditional) probabilities $p(A)$ and $p(B)$, we can calculate $p(A \mid B)$ from $p(B \mid A)$ [or vice versa]
  - Sometimes, prior probabilities are known as base rates

Bayes’ Theorem, The Sequel—Back To Bayes

- Theorem: $p(A \mid B) = p(B \mid A) \cdot p(A) / p(B)$
- A special case of Bayes’ Theorem relates conditional probabilities to both the probability that an event occurs and the probability that it doesn’t
- Assuming $p(A)$, $p(-A)$ [i.e., $p(S-A)$], and $p(B)$ are non-zero (so $p(B \mid A)$, $p(B \mid -A)$, and $p(A \mid B)$ are all defined), then:
  $$p(A \mid B) = \frac{p(B \mid A) \cdot p(A)}{p(B \mid A) \cdot p(A) + p(B \mid -A) \cdot p(-A)}$$
- How could we derive this formula?
  - Hint: $B = (B \cap A) \cup (B \cap -A)$
  - Hint: defn of conditional probability, $p(B \mid A) = p(B \cap A) / p(A)$
Bayes’ Theorem, The Sequel—Back To Bayes

- A special case of Bayes’ Theorem relates conditional probabilities to both the probability that an event occurs and the probability that it doesn’t.
- Assuming \( p(A), p(-A) \) [i.e., \( p(S-A) \)], and \( p(B) \) are non-zero (so \( p(B \mid A), p(B \mid -A), \) and \( p(A \mid B) \) are all defined), then:
  \[
p(A \mid B) = \frac{p(B \mid A) \cdot p(A)}{p(B \mid A) \cdot p(A) + p(B \mid -A) \cdot p(-A)}
  \]
- This means we can compute conditional probability \( p(A \mid B) \) if we know all the following:
  - \( p(A) \)—the prior probability of \( A \)
  - \( p(B \mid A) \)—the conditional probability of \( B \) given \( A \)
  - \( p(B \mid -A) \)—the conditional probability of \( B \) given \( -A \)

An Example

\[
p(A \mid B) = \frac{p(B \mid A) \cdot p(A)}{p(B \mid A) \cdot p(A) + p(B \mid -A) \cdot p(-A)}
\]
- There are two boxes. The first contains 2 green balls and 7 red balls; the second contains 4 green balls and 3 red balls.
- You select a ball at random—first, you select a box at random, and then, you select a ball at random from that box.
- If you select a red ball, what is the probability that you selected from the first box?
An Example

\[ p(A \mid B) = \frac{p(B \mid A) * p(A)}{p(B \mid A) * p(A) + p(B \mid \neg A) * p(\neg A)} \]

- There are two boxes. The first contains 2 green balls and 7 red balls; the second contains 4 green balls and 3 red balls.
- You select a ball at random—first, you select a box at random, and then, you select a ball at random from that box.
- If you select a red ball, what is the probability that you selected from the first box?
  - Let event \( E \) be the event of choosing a red ball, so \( \neg E \) is the event of choosing a green ball. Also, let \( F \) be the event of choosing from the first box, so \( \neg F \) is the event of choosing from the second box.
  - We’re looking for \( p(F \mid E) \). What’s \( p(E \mid F) \)? \( p(E \mid \neg F) \)? \( p(F) \)? \( p(\neg F) \)?

Another Example

\[ p(A \mid B) = \frac{p(B \mid A) * p(A)}{p(B \mid A) * p(A) + p(B \mid \neg A) * p(\neg A)} \]

- Suppose that one person in 100,000 has a rare disease for which there is a fairly accurate diagnostic test.
  - The test is correct 99.0% of the time when given to a person selected at random who has the disease
  - It is correct 99.5% of the time when given to a person selected at random who does not have the disease
- Should a person who tests positive be very concerned that he or she has the disease?
- What is the probability that…
  - … a person who tests positive has the disease?
  - … a person who tests negative doesn’t have the disease?