CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – W F 1:30pm
Lab – F 3:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• Questions about the last lecture? (Or the reading?)
• Reading: Ch. 1.1-1.4 in our textbook
• Lecture, not lab, in 3:30pm class meeting today

This week only: We will meet here, SP 105, at 3:30 on Friday!

• Please be sure you have a working cs.vassar account!

• Please email me (as per Assignment from last lecture) by the end of the day on Saturday!
  – Note: my preferred email is eaaron@cs.vassar.edu (not eraaron@vassar.edu)
  – Thank you to those who have emailed me already!

My apologies if I haven’t replied to your email yet—I will do so as soon as I can!
But before the set up, a little math review…

• What’s the formula for the sum of the first $n$ positive integers?

• How do we prove it?

• How would we write a simple Scheme function `sum-one-to-n` that takes input $n$ and returns the sum from 1 to $n$?

Sum Scheme

• A definition of summation from 1 to $n$:
  – If $n = 1$, the sum is 1
  – If $n > 1$, the sum is $n + \text{[the sum from 1 to (n-1)]}$

• How does this definition relate to…
  – Our inductive proof?
  – Our Scheme code?

```
(define sum-one-to-n
  (lambda (n)
    (if (= n 1)
        1
        (+ n (sum-one-to-n (- n 1))))))
```
Mathematical In(tro)duction

• Mathematical induction is a proof technique to prove statements of the form
  – For every natural number \( n \) (or perhaps every integer greater than some fixed value, such as “all integers greater than -4” or “all integers greater than 2”)
  – … some statement in terms of \( n \) is always true

• Intuitive idea: Proving that all the dominos will fall
  (1) First, make sure the first domino falls
  (2) Then, make sure they’re all set up such that if you look at any domino in the chain—say, the \( k \)’th domino, for any \( k \)—then if the \( k \)’th domino falls, then the \( (k+1) \)’st will fall

• This is enough to prove that every domino falls! Think about it a bit…

There are other flavors of induction, but we’ll focus on this one for now

Mathematical In(tro)duction, cont.

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• This is enough to prove that every domino falls! Think about it a bit…
  – The first one falls, because we prove that directly with (1)
  – The second one falls, because we consider (2) with \( k = 1 \)—it says if the first domino falls, so does the second
  – Similarly, the third one falls, because we consider (2) with \( k = 2 \)…
  – Similarly, the fourth one falls… (is it clear why?)
  – And we can show that all of them fall! How?
Mathematical Induction, cont.

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  (1) First, make sure the first domino falls
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• This is enough to prove that every domino falls!
  – The first one falls, because we prove that directly with (1)
  – The second one falls, because we look at (2) with k = 1…

• And we can show that all of them fall! How?
  – One way to think of it: Show that it can’t be any other way! (Proof by contradiction)
  – What if one didn’t fall? We don’t know which, it could be any, so we’ll call it domino d that (for argument’s sake, for now) didn’t fall
  – Then we’d know… what?

Mathematical Induction, cont.

• Intuitive idea: Proving that all the dominos will fall
  (1) First, make sure the first domino falls
  (2) Then, make sure they’re all set up such that if you look at any domino in the chain—say, the k’th domino, for any k—then if the k’th domino falls, then the (k+1)’st will fall

• And we can show that all of them fall! How?
  – One way to think of it: Show that it can’t be any other way! (Proof by contradiction)
  – What if one didn’t fall? We don’t know which—it could be any, so we’ll call it domino d that (for argument’s sake, for now) didn’t fall
  – Then we’d know that domino d-1 didn’t fall either! See (2), above—if d-1 did fall, d would have fallen; but d didn’t; so d-1 couldn’t have fallen
  – By the same reasoning d-2 didn’t fall! Or d-3! Or…
  – … Do you see how this results in a contradiction?
Mathematical Induction, cont.

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  1. First, make sure the first domino falls
  2. Then, make sure they’re all set up such that if you look at any domino in the chain—say, the k’th domino, for any k—then if the k’th domino falls, then the (k+1)’st will fall
- And we can show that all of them fall! How?
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  - By the same reasoning d-2 didn’t fall! Or d-3! Or…
  - … Or the first domino—the first domino must not have fallen either! But by (1)—which we know to be true, the first one did fall. Contradiction!

Mathematical Induction, cont.

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  1. First, make sure the first domino falls
  2. Then, make sure they’re all set up such that if you look at any domino in the chain—say, the k’th domino, for any k—then if the k’th domino falls, then the (k+1)’st will fall
- And we can show that all of them fall! How?
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Inductive Proofs

- Our inductive proof, like all inductive proofs, has four parts
  1. Write down what we’re trying to prove
     - Be sure to describe the variables! (e.g., “For every number $n$ greater than 1…”)
  2. Prove the base case—the first domino falls
  3. Write the inductive hypothesis—assume domino $k$ falls
  4. Prove your inductive case—using the inductive hypothesis, show that domino $k+1$ falls too

Every step in this can be subtle, or require some thought—we’ll see examples as the course goes on!

Equational Proof Style for Inductive Proofs

- Here’s a presentation of the inductive case of the proof, using our equational proof style
  - We’re trying to prove two expressions are equal:
  - Every step in the proof asserts an equality (and gives a reason!)

\[
\begin{align*}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k + 1) \\
&= \frac{k(k+1)}{2} + (k + 1) \\
&= \frac{k(k+1) + 2(k + 1)}{2} \\
&= \frac{k + 1}{2} \cdot (k + 1 + 1) \\
&= \frac{(k + 1)(k + 1 + 1)}{2}
\end{align*}
\]

This information will also be in a document on the course website!
Equational Proof Style for Inductive Proofs

- This can be a useful and compact style for some kinds of proofs
  - Good for the kind of reasoning done in these inductive proofs
  - For some proofs, it may be more natural not to use this style—we’ll see plenty of non-equational style proofs in class this semester too!
- General point of style: Avoid working on (i.e., changing) both sides of an equation

Here, the proof can be seen as having only the right side of an equation change in each step—the left side stays the same. (Note that the first left-hand side isn’t even rewritten!)

Also note that on each line, the given reason is a justification for the equality on that line.

Another Example Induction

- Show that the sum of the first \( n \) odd numbers is \( n^2 \)
- Proof by induction! Go through the steps
  - Write down what we’re trying to prove
  - What’s our base case?
  - What’s our inductive hypothesis?
  - How do we use our inductive hypothesis to prove our inductive case?

Getting started with induction can feel a little getting started with recursion—like there’s some magic to it! (“Wait… that’s the whole recursive program?”)

But as with recursion, you learn induction by thinking through several examples. We’ll see more of it as the course goes along!