CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – W F 1:30pm
Lab – F 3:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW2 out, due Sept. 27
  – (Please get an early start! HW3-Lookahead may be out soon, due to scheduling regarding the last days of classes before break!)
  – IMPORTANT: As emailed, HW2 re-posted today with clarification on Ex. 3—now more specific on details to include in your answer
• HW1 grading update
• Changes in Office Hours this week:
  – No Office Hours tomorrow (due to Holiday)
  – Extra Hours: today, noon-1pm; Thursday, Sept. 21, 6-6:30pm
• Please read Ch.2.1, 2.2.2, 2.3.1, 2.4, 2.5, and 2.7.1 in your textbook
  – You can skip section 2.3.2, but people interested in databases might want to read it anyway
• Lab1 due by the end of the day, Thurs., Sept. 21
  – As always, must be checked off and submitted online

(note: graph images in this lecture are from the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest, and Stein)
Binary Relations

- A *relation*, intuitively enough, expresses the relation between elements of various sets. More formally...

- For sets A and B, a *binary relation from A to B* (or *over* $A \times B$) is a subset of $A \times B$. *(That's the Cartesian product of A and B)*
  - i.e., it is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$
  - … thus, it relates elements of A to elements of B
  - Note: If a relation R is a subset of $A \times A$, we say it is a relation *over* A

- More generally, for sets $A_1, \ldots, A_n$, an $n$-place relation over $A_1, \ldots, A_n$ is a subset of the set product $A_1 \times \ldots \times A_n$

Directed Graphs

- One way of representing a binary relation is a *directed graph*, which indicates ordered relations between elements

![Directed Graph Examples](image)

(a) and (c) above are directed graphs; (b) is undirected. More on that soon!

- A directed graph G is a pair $(V, E)$ where V is a finite set of *vertices* (singular: *vertex*), and E is a binary relation on $V \times V$
  - E is called the *edges* (or *edge set*) of graph G

- Exercise: For graph (a), what are the sets V and E?
Properties of Relations

• Some useful properties of relations! Definitions:
  – A relation R over a set A is reflexive if for all a ∈ A, (a,a) ∈ R
    • What would a reflexive relation be over the set {1,2,3}? (If it’s clear in context what the set A is, we might simply say that the relation R is reflexive, in that context)
  – A relation R is symmetric if whenever (a,b) ∈ R, (b,a) ∈ R
  – A relation R is transitive if whenever (a,b) ∈ R and (b,c) ∈ R, then (a,c) ∈ R
• Exercise: Consider the following relations over {1,2,3,4}
  • R1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4, 4)}
  • R2 = {(1,1), (1,2), (2,1)}
  • R3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
  • R4 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
  – Which of these are reflexive? symmetric? transitive?

More Vocabulary and Properties of Relations

• For relation R:
  – Domain \( \text{dom}(R) = \{a \mid \exists b \text{ s.t. } (a,b) \in R\} \) — that is, the domain is all elements that are the first element of a pair in R
  – Range \( \text{range}(R) = \{b \mid \exists a \text{ s.t. } (a,b) \in R\} \) — that is, the range is all elements that are the second element of a pair in R
• Recall that relations are sets of n-tuples; binary relations are sets of ordered pairs. So, applying ideas from subsets:
  – A relation R could be a subrelation of a relation S
  – The empty relation has no elements—it is the same as \( \emptyset \)
• Exercises:
  – What is \( \text{dom}(A \times B) \)? What is \( \text{range}(A \times B) \)?
  – What does it mean if relations R and S are disjoint? If A = {1,2,3}, what is an example of disjoint relations over A?
  – Consider a relation R and its inverse \( R^{-1} \). How do \text{dom}(R) and \text{range}(R) relate to \text{dom}(R^{-1}) and \text{range}(R^{-1})?
Digression: Undirected Graphs

- There are also undirected graphs
  - An undirected graph is a pair $G=(V,E)$ where
    - $V$ (vertices) is a finite set, and
    - $E$ (edges) is a set of unordered pairs on $V \times V$

- What are $V$ and $E$ for undirected graph (b), above?

Digression: Weighted Graphs

- Weighted graph—good for representing things like maps or networks
  - Weighted graphs can be directed or undirected; the one below is undirected

A weighted, undirected graph representing distances between Romanian cities, from *Artificial Intelligence: A Modern Approach* by Russell and Norvig
(the above jpeg version is taken from http://www.emunix.emich.edu/~sverdlik/COSC461)
Still A Digression:  
Still More Graph Vocabulary

• What kinds of questions do you think might be interesting to answer about graphs?

• Some common graph vocabulary:
  – If \((u,v)\) is an edge, we say vertex \(v\) is adjacent to vertex \(u\)
  – The degree of a vertex in an undirected graph is the number of edges incident on it; in directed graphs, vertices have in degrees and out degrees
  – An undirected graph is connected if…; its connected components are…
  – A directed graph is strongly connected if…

Subrelations and Subgraphs

• We can also talk about subgraphs of a graph
  – In general, for a graph \(G = (V,E)\), graph \(G' = (V',E')\) is a subgraph of \(G\)
    iff \(V' \subseteq V\) and \(E' \subseteq E\)
  – i.e., if the vertices and edges of \(G'\) are subsets of the vertices and edges of \(G\)
  – It can be useful to talk about a subgraph induced by a particular subset \(V'\) of \(V\)—that is the subgraph containing all vertices in \(V'\) and all edges with both endpoints in \(V'\)

• Exercises:
  – In graph (c) below, what is the subgraph induced by \(\{1,2,3\}\)? How would you write it as a relation?