CS 145 – Foundations of Computer Science

Professor Eric Aaron

Lecture – W F 1:30pm
Lab – F 3:30pm

Lecture Meeting Location: SP 105
Lab Meeting Location: SP 309

Business

• HW4 extended: Now due Oct. 31 / Nov. 1 instead of Oct. 26 / Oct. 27 (see assignment sheet for deadlines)

• Reading: Ch.4.1-4.6
  – Our coverage of the material will be different from that in the textbook, but it’s good to see the textbook’s presentation, as well

• Reading: Prof. Hunsberger’s document “The Natural Numbers, Induction, and Numeric Recursion”
  – Posted on the Additional Notes / Readings page of the CS145 website

• Lab on Friday this week
  – Lab due by end of the day Sunday, Oct. 29

Note the unusual “due date” for Lab!
Business, pt. 2: A note about how HWs are graded

- Just to be clear, homeworks are evaluated on criteria including the following:
  - Are the necessary (and sufficient) ideas present for a full-credit answer?
  - Does the written answer demonstrate full command of those ideas?
  - Does the written answer demonstrate command of appropriate formal notation for rigorous expression of the relevant ideas?

So, unless explicitly instructed to do otherwise, answers should include the ideas, a thorough explanation, and appropriate formal notation for those ideas / explanation!

Business, pt. 3

- Visiting Speaker today!
- Presentation about the AIT-Budapest study abroad program / JYA option for Computer Science
  - Title: CS Study Abroad in Budapest
  - Speaker: Gábor Bojár, Founder and Professor of IT Entrepreneurship, AIT-Budapest program
  - Date: Wednesday, Oct. 25
  - Place: SP 105
  - Time: 3:15pm
  - Snacks / Refreshments: Will be served!
A Recursively Defined Set: The Natural Numbers

- As suggested by all of our inductive proofs about numbers, there is also a recursive definition of the natural numbers.
- The Peano axioms are conventionally taken as a definition of the natural numbers (here, let $N$ stand for the natural numbers):
  1. There exists a number $0$ s.t. $0 \in N$
  2. Every natural number $n$ has a natural number successor, denoted by $S(n)$
  3. There is no $n$ in $N$ s.t. $S(n) = 0$
  4. Distinct natural numbers have distinct successors: if $a \neq b$, then $S(a) \neq S(b)$
  5. Let $P$ be a property of the natural numbers such that:
     - $P(0)$ holds
     - For every $a$ in $N$, if $P(a)$ holds, then $P(S(a))$ holds

If both of those conditions are true, then $P(n)$ holds for all $n$ in $N$.

Axioms 1 and 2 give the recursive construction of the elements of $N$. The other Axioms are properties of $N$. Axiom 5 is sometimes called the Axiom of Induction.

Peano Examples

- The Peano axioms:
  1. There exists a number $0$ s.t. $0 \in N$
  2. Every natural number has a natural number successor, denoted by $S(n)$
  3. There is no $n$ in $N$ s.t. $S(n) = 0$
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     - $P(0)$ holds
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If both of those conditions are true, then $P(n)$ holds for all $n$ in $N$.

- Exercises and examples:
  - Are there two different ways to write the number 2? (Hint: Prove the following Theorem)
  - Theorem: Every natural number $n$ has one of two forms, either $n=0$ or $n=S(m)$ for some $m$.
    - Proof: By induction!

(See Hunsberger’s “The Natural Numbers...” document)
Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…
  – What would the base case(s) be?
  – What would the inductive case(s) be?

Note: We proved that every natural number is either 0 or S(m) for some m. How can that help us in this recursive definition?

More Addition

• If the Peano axioms define the natural numbers…
  – How could we define the addition function?
  – Hint: Recursively! Because our definition of the numbers is recursive…

• Definition of addition:
  1. Case z=0 -- For all n in N, n + 0 = n
  2. Case z=S(m) -- For all n in N, z = S(m) for some m in N:
     n + S(m) = S(n + m)

• Let’s prove something with that definition!
  – Claim: This addition function is associative (i.e., a + (b + c) = (a + b) + c, for all a,b,c in N)
  – Proof: ??
Proof: Associativity of Addition

• Definition of addition:
  1. Case z=0 -- For all n in N, n + 0 = n
  2. Case z=S(m) -- For all n in N, z = S(m) for some m in N: n + S(m) = S(n + m)

• Let’s prove something with that definition—associativity!
  – To prove: For all a, b, c in N, a + (b + c) = (a + b) + c
  – Proof: Prove by induction on c.
  – Let P(c) be the proposition: For all a, b in N, a + (b + c) = (a + b) + c
  – Base—c=0.
    – P(0) is the proposition: For all a, b in N, a + (b + 0) = (a + b) + 0
    – To prove: P(0)—for arb’ly chosen a, b, show a + (b + 0) = (a + b) + 0
      • Start with the left hand side: a + (b + 0) = a + b, because (b + 0) = b (by equation 1 of the definition of addition, with b in place of n); then, a + b = (a + b) + 0, also by equation 1, with (a + b) in place of n. Thus, a + (b + 0) = (a + b) + 0, completing the proof of the base case.