Exercise 1

In many cases, it's possible to simplify a complicated summation by replacing it with a simple expression. For example, in lecture we proved that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. (I recommend reviewing this proof.) In this exercise, you'll prove that you can simplify a different summation.

Recall that the Fibonacci sequence is the famous (rabbit-counting) sequence defined by this function $F$:

\[
F(0) = 0 \\
F(1) = 1 \\
F(n) = F(n - 1) + F(n - 2) \quad \text{whenever } n \geq 2
\]

Thus, the first few Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Now consider the sum of the squares of Fibonacci numbers. Specifically, for $n \geq 1$, think about the sum $F(0)^2 + F(1)^2 + \cdots + F(n)^2$. Find a simple expression for this summation and prove by induction that your formula is correct.
Exercise 2

Recall the definition of the addition function, seen in class, where $S$ stands for the "successor" function:

(1) $x + 0 = x$ for all natural numbers $x$

(2) $x + S(y) = S(x + y)$ for all natural numbers $x$ and $y$

Prove that the addition function defined above is commutative. In other words, prove the following:

$x + y = y + x$ for all natural numbers $x$ and $y$

In your proof, only use rules 1 and 2 above and these properties:

(3) $0 + x = x$ for all natural numbers $x$

(4) $x + S(y) = S(x) + y$ for all natural numbers $x$ and $y$