Due November 5, 1:30 p.m.

Please follow the form of the proofs by induction in the handouts you have received. That is: be sure to explicitly write down the proposition \( P(n) \) you are using; explicitly write down the base case and inductive case; and carefully show how the lefthand side can be transformed into the righthand side via a clear sequence of well-justified steps. Full credit will only be given to well-organized, concisely and neatly written proofs.

**Problem 1**

Recall the definition of the addition function, seen in class, where \( S \) stands for the “successor” function:

1. \( x + 0 = x \) for all natural numbers \( x \)
2. \( x + S(y) = S(x + y) \) for all natural numbers \( x \) and \( y \)

Additional properties for addition are:

3. \( 0 + x = x \) for all natural numbers \( x \)
4. \( x + S(y) = S(x) + y \) for all natural numbers \( x \) and \( y \)

Prove that the addition function defined above is commutative. In other words, prove the following:

\[ x + y = y + x \]

for all natural numbers \( x \) and \( y \)

Your proof should use only rules 1–4 above.
Problem 2

Recall definition of multiplication over the natural numbers:

(5) \( x \cdot 0 = 0 \) for all natural numbers \( x \)

(6) \( x \cdot S(y) = (x \cdot y) + x \) for all natural numbers \( x \) and \( y \)

Prove inductively that multiplication distributes over addition, using the definitions for addition (from Problem 1) and multiplication (above). In other words, prove that

\[
(x + y) \cdot z = (x \cdot z) + (y \cdot z)
\]

for all natural numbers \( x, y \), and \( z \).

• **Hint:** Let \( P(z) \) be the proposition that \( (x + y) \cdot z = (x \cdot z) + (y \cdot z) \) for all natural numbers \( x \) and \( y \). Then use induction to prove that \( P(z) \) holds for all natural numbers \( z \).

• **Note:** You may freely use the associative and commutative properties of addition in your proof. However, you must show where in your proof you need these properties. The easiest way to do this is to make sure that your expressions are all fully parenthesized. When you need to change the parentheses, that’s when you need associativity. When you need to re-order terms, that’s when you need commutativity. Also, for convenience, you may do a bunch of applications of these two properties in a single step.