Exam 2 Practice

Exam 2 will be open-book and open-notes. The following problems are examples of the topics and types of problems you will see on the exam. However, the practice problems may not reflect the length of the exam or its distribution of question types.

Review

This is a review of some of the major concepts we've covered since Exam 1.

Relations

- Relations express properties involving two or more entities.
- The empty relation has no elements: \( R = \{\} = \emptyset \).
- The Cartesian product \( A \times B \) is the set of all ordered pairs \((a, b)\) such that \(a \in A\) and \(b \in B\).
- Any binary relation \( R \) over sets \( A \) and \( B \) can be described as a set of ordered pairs, which will always be a subset of \( A \times B \).
- Thus we can say \((x, y) \in R\) or \(x R y\), which is like how we've always written relations like “less than” \((x < y)\) or “element of” \((x \in Y)\).
- Taking the inverse of a binary relation reverses the order of each ordered pair: \( R^{-1} = \{(b, a) \mid (a, b) \in R\} \).
- Reflexive relations: If \( R \) is a binary relation over set \( A \), for all \( a \in A \), \((a, a) \in R\).
- Symmetric relations: whenever \((a, b) \in R\), \((b, a) \in R\).
- Transitive relations: whenever \((a, b) \in R\) and \((b, c) \in R\), then \((a, c) \in R\).
- Equivalence relations are reflexive, symmetric, and transitive.
- Partitions divide a set \( A \) into subsets \( B_i \), so every element of \( A \) belongs to exactly one subset.
  - The subsets must exhaust \( A \) (i.e., use every element of \( A \))
  - The subsets must be pairwise disjoint (i.e., not share any elements)
  - Every partition corresponds to an equivalence relation “in the same cell”
• Ordering relations define an order for elements:
  – All ordering relations are transitive and antisymmetric.
  – An inclusive ordering relation is reflexive (like \( \leq \)), but a strict ordering relation is not reflexive (like \(<\)).
  – A partial ordering of a relation \( R \) over a set \( A \) is reflexive, transitive, and antisymmetric.
  – A total ordering is a kind of partial ordering, where the relation \( R \) specifies an order for all elements of \( A \) (we say the ordering is “complete over \( A \)”).

Functions

• Functions are a kind of relation. Since functions are relations, all operations we applied to relations can be applied to functions.

• One-place (or unary) function: A binary relation \( R \), from \( A \) to \( B \), such that for all \( a \in A \), there is exactly one \( b \in B \) for which \( (a, b) \in R \). That is, a function \( f : A \to B \) maps every element of \( A \) to exactly one \( B \).

• Partial function: a binary relation \( R \), from \( A \) to \( B \), such that for all \( a \in A \), there is at most one \( b \in B \) for which \( (a, b) \in R \).

• Injective functions: whenever \( x \neq y \), \( f(x) \neq f(y) \).

• Surjective functions: for all \( b \in B \), there exists \( a \in A \) with \( f(a) = b \).

• Bijective functions: both injective and surjective.

• Principle of equinumerosity: For finite sets \( A \) and \( B \), \( |A| = |B| \) iff there is a bijective function \( f : A \to B \).

• Principle of comparison: For finite sets \( A \) and \( B \), \( |A| \leq |B| \) iff there is an injective function \( f : A \to B \).

• Pigeonhole Principle: For finite sets \( A \) and \( B \), if \( |A| > |B| \), then no function \( f : A \to B \) is injective.

Proof methods

• Writing proofs based on first-order logic definitions.

• Proving a conditional: \( \varphi \Rightarrow \psi \) is true in all cases except when \( \varphi \) is true yet \( \psi \) is false. To prove it directly, assume \( \varphi \) and show that \( \psi \) must still hold.

• Proving a biconditional: \( \varphi \iff \psi \) is true whenever \( \varphi \) and \( \psi \) have the same truth value. To prove it directly, prove \( \varphi \Rightarrow \psi \) and \( \psi \Rightarrow \varphi \).
• Negating quantified claims
  – \(\neg \exists\) (“It’s not the case that there is…”) becomes \(\forall \neg\) (“For every…it is not the case”)
  – \(\neg \forall\) (“It’s not the case that every…”) becomes \(\exists \neg\) (“There is…such that not…”)

• Proof by induction:
  – **Base case:** verify the statement is true for the basis of the definition (like \(n = 0\) for the natural numbers)
  – **Induction case:** Assume the statement is true for an arbitrary value (induction hypothesis) like \(n\) and use that to show the statement is also true for the next ordered step (induction goal) like \(n + 1\).

• The Peano axioms give a recursive definition of the set of natural numbers (\(\mathbb{N}\)) that gives us the means to write inductive proofs about them. We can similarly define other sets, e.g., the set of lists or the set of all propositional logic statements.
Problem 1: Properties of Relations

For this problem, we restrict attention to binary relations over some set $A$ (i.e., relations from $A$ to $A$). We have looked at the properties of a relation being reflexive, symmetric, and transitive:

- A binary relation $R$ over a set $A$ is reflexive if and only if for every $x \in A$, $(x, x) \in R$.
- A binary relation $R$ over a set $A$ is symmetric if and only if whenever $(x, y) \in R$, then so is $(y, x) \in R$.
- A binary relation $R$ over a set $A$ is transitive if and only if whenever $(x, y) \in R$ and $(y, z) \in R$, then so is $(x, z) \in R$.

Here’s a new property:

A binary relation $R$ over a set $A$ is serial if and only if for every $x \in A$, there is some $y \in A$ such that $(x, y) \in R$.

For example, the relation, $\{(1, 2)\}$, over the set $A = \{1, 2, 3\}$ is not serial because (for example) $2 \in A$, but there is no $y \in A$ such that $(2, y)$ is in the relation. On the other hand, the relation

$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

over the set $A = \{1, 2, 3\}$ is serial since for each $x \in A$ there is at least one $y \in A$ such that $(x, y)$ is in the relation. In this case, each $x \in A$ has three pairs of the form $(x, y)$ in the relation.

a. Give an example of a relation $R$ over the set $\{1, 2, 3\}$ that is serial, but not reflexive.

b. TRUE or FALSE (Circle one). If $f : A \to A$ is a function, then it must be a serial relation.
c. Prove that if a binary relation $R$ over some set $A$ is reflexive, then it must also be serial.

d. Prove that if a binary relation $R$ over some set $A$ is serial, symmetric, and transitive, then it must also be reflexive.
Problem 2: Relations, Proof, and Logic

Consider the following proof:

| THEOREM | If a binary relation $R$ over the set $A$ is not reflexive, then it is irreflexive. |
| PROOF | Since $R$ is not reflexive, there must be at least one $x \in A$ such that $x \mathbin R x$ does not hold. Since the choice of $x$ was arbitrary, we must therefore have that for any $x \in A$, $x \mathbin R x$ does not hold. Thus $R$ is irreflexive. |

This proof is incorrect because the statement of the theorem does not match the statement that's proven.

a. Translate the statement of the theorem into first-order logic.

$$Reflexive: \forall a \in A . a \mathbin R a$$

$$Irreflexive: \forall a \in A . a \not\mathbin R a$$

b. Translate the theorem that was actually proven into first-order logic.

c. Explain why the statement of the theorem and the statement that was proven are not the same thing.
Problem 3: Functions

Consider the following Euler diagram: 6 points

Below is a list of purported functions. For each one, write its number where it belongs on the diagram. No justification is necessary. To get you started, we’ve shown where the first two go!

1. \( f: \mathbb{N} \to \mathbb{N} \) defined as \( f(n) = 137 \)
2. \( f: \mathbb{N} \to \mathbb{N} \) defined as \( f(n) = -137 \)
3. \( f: \mathbb{N} \to \mathbb{N} \) defined as \( f(n) = n^2 \)
4. \( f: \mathbb{Z} \to \mathbb{N} \) defined as \( f(n) = n^2 \)
5. \( f: \mathbb{N} \to \mathbb{Z} \) defined as \( f(n) = n^2 \)
6. \( f: \mathbb{Z} \to \mathbb{Z} \) defined as \( f(n) = n^2 \)
7. \( f: \{0, 1, 2\} \to \{3, 4\} \), where \( f \) is some surjective function.
8. \( f: \{\text{breakfast, lunch, dinner}\} \to \{\text{shakshuka, soondubu, maafe}\} \), where \( f \) is some injection.
Problem 4: Recursive Functions in Racket

Define a Racket function, called \texttt{fracs}, that takes two integer inputs, \texttt{n} and \texttt{total}. It should return as its output a list of fractions, as illustrated below:

\begin{verbatim}
> (fracs 4 5)
'(1/4 2/3 3/2 4)
> (fracs 6 7)
'(1/6 2/5 3/4 4/3 5/2 6)
\end{verbatim}

Notice that the value of \texttt{n} is the value of the first denominator in the output list, and that subsequent denominators decrease by 1 each time. The sum of the numerator and denominator of any fraction in the output list is \texttt{total}.

- It may help to write down the recursive function call. Assuming it works properly, what should the recursive function call return as its output?
- The value of \texttt{total} should stay the same across the recursive function calls.
Problem 5: Proof by Induction

Multiplication (⋅) can be defined recursively, using a successor function $S$:

1. $x \cdot 0 = 0$ for all $x \in \mathbb{N}$
2. $x \cdot S(y) = (x \cdot y) + x$ for all $x, y \in \mathbb{N}$

Use this definition to prove inductively that multiplication is associative. In other words, prove that

$$x \cdot (y \cdot n) = (x \cdot y) \cdot n$$

for all natural numbers $x, y, n$.

Note: For this problem, you can assume that addition is associative and commutative, and that multiplication distributes over addition.