The final exam is cumulative, though it may emphasize material covered since Exam 2. It will be open-book and open-notes. The following are examples of the topics and types of problems you may see on the exam.

Set theory & proofs

See Chapter 1.

- **Subset**: For any sets $A$ and $B$, $A \subseteq B$ iff for every $x$, if $x \in A$, then $x \in B$.
- **Identity**: For any sets $A$ and $B$, $A = B$ if both $A \subseteq B$ and $B \subseteq A$.
- **Proper subset**: For any sets $A$ and $B$, $A \subset B$ iff $A \subseteq B$, but $A \neq B$.
- **Set-builder notation**: $\{x \in \mathbb{N}_0 \mid x < 10\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- **Sets** $A$ and $B$ are **disjoint** iff there is no $x$ such that $x \in A$ and $x \in B$.
- **Intersection**: For any sets $A$ and $B$, for all $x$, $x \in (A \cap B)$ iff $x \in A$ and $x \in B$.
- **Union**: For any sets $A$ and $B$, for all $x$, $x \in (A \cup B)$ iff $x \in A$ or $x \in B$.
- **Difference**: For any sets $A$ and $B$, for all $x$, $x \in (A - B)$ iff $x \in A$ but $x \notin B$.
- **Power set**: Collects all subsets of a particular set:
  - For any set $A$, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$.
  - $\mathcal{P}(A)$ always include $\emptyset$, the empty set, because the empty set is a subset of every set.
  - $\mathcal{P}(A)$ always includes $A$ because $A \subseteq A$.
  - If $A$ contains 1, $\mathcal{P}(A)$ doesn't contain 1, but it contains the singleton set $\{1\}$.
- The **Cartesian product** $A \times B$ is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. 
Exercise: Cartesian product

Describe a deck of playing cards as the Cartesian product of two sets.

Exercise: Power set proof

Let $A$ and $B$ be arbitrary sets. Prove that $A \in \mathcal{P}(B)$ if and only if $A \cap B = A$. 
Propositional & first-order logic

See Chapters 8 and 9.

- A proposition is a claim with a truth value, like “2 + 5 = 7” (true) or “The capital of New York is New York City” (false).

- We usually represent propositions using individual, lowercase letters, e.g., p, q, r, s, ...

- Propositional logic relations:
  - Negation ("not"), written as \( \neg \varphi \),
  - Conjunction ("and"), written as \( \varphi \land \psi \),
  - Disjunction ("or"), written as \( \varphi \lor \psi \),
  - Conditional or implication ("if...then"), written as \( \varphi \Rightarrow \psi \), and
  - Biconditional, bidirectional implication, or equivalence ("if and only if"), written as \( \varphi \Leftrightarrow \psi \).

- You can think of the logical relations as functions, which take propositions as input and return a truth value as output (\( \lor : E \rightarrow \{ \top, \bot \} \)).

- We can characterize propositions as:
  - Tautology: a proposition that is true regardless of the truth values of its components, e.g., \( p \lor \neg p \).
  - Contradiction: a proposition that is false regardless of the truth values of its components, e.g., \( p \land \neg p \).
  - Contingent: a proposition whose truth depends on the truth values of its components, e.g., \( p \lor q \).
  - Satisfiable: a proposition that is true for at least one possible combination of truth values for its components. A contradiction is never satisfiable. Contingent propositions or tautologies are satisfiable.

- Logical equivalence: two statements are logically equivalent if they have the same truth table (that is, if they evaluate to the same truth result for all possible truth values of the propositions they use).

- The converse and inverse of an implication are logically equivalent, because they have the same truth table.

  - The converse of an implication reverses the order of two components.
    \[ p \Rightarrow q \text{ becomes } q \Rightarrow p \]
- The inverse of an implication negates the two components.
  \[ p \implies q \text{ becomes } \neg p \implies \neg q \]

- The contrapositive of an implication reverses the order of the two components and negates both of them.
  \[ p \implies q \text{ becomes } \neg q \implies \neg p \]

An implication and its contrapositive are logically equivalent, because they have the same truth tables.

- First-order logic extends propositional logic to add quantifiers and predicates.

- Quantifiers:
  - \( \exists x \) meaning “there is” (or “there are”, “there exists”)
  - \( \forall x \) meaning “for all” (or “for every”, “for any”)

- Predicates (that is, properties):
  - Poet(\( x \)) asserts that \( x \) is a poet.
  - Dog(\( y \)) asserts that \( y \) is a dog.
  - Likes(\( x, y \)) asserts that \( x \) likes \( y \).

- We often write proofs based on first-order logic definitions, but you shouldn’t use logical syntax inside your proofs. See the guide to proofs on relations.

- Proving a conditional: \( \varphi \implies \psi \) is true in all cases except when \( \varphi \) is true yet \( \psi \) is false. To prove it directly, assume \( \varphi \) and show that \( \psi \) must still hold.

- Proving a biconditional: \( \varphi \iff \psi \) is true whenever \( \varphi \) and \( \psi \) have the same truth value. To prove it directly, prove \( \varphi \implies \psi \) and \( \psi \implies \varphi \).

- Negating quantified claims
  - \( \neg \exists \) ("It’s not the case that there is...") becomes \( \forall \neg \) ("For every...it is not the case")
  - \( \neg \forall \) ("It’s not the case that every...") becomes \( \exists \neg \) ("There is...such that not...")
Exercise: English to FOL

Express the following sentences in first-order logic (FOL), using appropriate quantifiers and predicates. Note any limitations you encounter.

a. “All wookiees have fur.”

b. “There is an ocean on Europa.”

c. “Every planet orbits some star.”

Note: Not necessarily the same star!
Relations & functions

See Chapters 2 and 3 and the Guide to Proofs about Relations.

- **Relations** express properties involving two or more entities.
- The **empty relation** has no elements: \( R = \{ \} = \emptyset \).
- Any binary relation \( R \) over sets \( A \) and \( B \) can be described as a set of ordered pairs, which will always be a subset of \( A \times B \).
- Thus we can say \((x, y) \in R \) or \( x \mathrel{R} y \), which is like how we've always written relations like “less than” \( (x < y) \) or “element of” \( (x \in Y) \).
- Taking the inverse of a binary relation reverses the order of each ordered pair: \( R^{-1} = \{ (b, a) \mid (a, b) \in R \} \).
- **Reflexive** relations: If \( R \) is a binary relation over set \( A \), for all \( a \in A \), \((a, a) \in R \).
- **Symmetric** relations: Whenever \((a, b) \in R \), \((b, a) \in R \).
- **Transitive** relations: Whenever \((a, b) \in R \) and \((b, c) \in R \), then \((a, c) \in R \).
- **Equivalence** relations are reflexive, symmetric, and transitive.
- **Partitions** divide a set \( A \) into subsets \( B_i \), so every element of \( A \) belongs to exactly one subset.
  - The subsets must **exhaust** \( A \) (i.e., use every element of \( A \))
  - The subsets must be **pairwise disjoint** (i.e., not share any elements)
  - Every partition corresponds to an equivalence relation “in the same cell”
- **Ordering** relations define an order for elements:
  - All ordering relations are transitive and antisymmetric.
  - An **inclusive** ordering relation is reflexive (like \( \leq \)), but a **strict** ordering relation is not reflexive (like \(< \)).
  - A **partial** ordering of a relation \( R \) over a set \( A \) is reflexive, transitive, and antisymmetric.
  - A **total** ordering is a kind of partial ordering, where the relation \( R \) specifies an order for all elements of \( A \) (we say the ordering is “complete over \( A \)”)
- **Functions** are a kind of relation. Since functions are relations, all operations we applied to relations can be applied to functions.
- **Injective** functions: Whenever \( x \neq y \), \( f(x) \neq f(y) \).
• Surjective functions: For all \( b \in B \), there exists \( a \in A \) with \( f(a) = b \).

• Bijective functions: Both injective and surjective.

• Principle of equinumerosity: For finite sets \( A \) and \( B \), \(|A| = |B|\) iff there is a bijective function \( f: A \to B \).

• Principle of comparison: For finite sets \( A \) and \( B \), \(|A| \leq |B|\) iff there is an injective function \( f: A \to B \).

• Pigeonhole Principle: For finite sets \( A \) and \( B \), if \(|A| > |B|\), then no function \( f: A \to B \) is injective.
**Exercise: Bijections and cardinality**

Two sets $A$ and $B$ have the same cardinality ($|A| = |B|$) if there is a bijective function $f: A \to B$.

The *perfect squares* are the squares of all the positive integers. Let $S = \{1, 4, 9, 16, \ldots\}$ be the set of perfect squares, and let $E = \{2, 4, 6, 8, \ldots\}$ be the set of positive even integers. Show that $|S| = |E|$. You don't need to prove that your function is a bijection.

**Exercise: Injective function proof**

Imagine you have a function $f: A \to B$ from some set $A$ to some set $B$. We can use $f$ to construct a new function called the *lift of $f$*, denoted $\text{lift}_f$, from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Specifically, $\text{lift}_f: \mathcal{P}(A) \to \mathcal{P}(B)$ is defined as follows:

$$\text{lift}_f(S) = \{y \mid \exists x \in S. f(x) = y\}$$

Let $A$ and $B$ be sets. Prove that if $f: A \to B$ is injective, then $\text{lift}_f$ is injective.
Induction & recursion

See Chapter 4, the Guide to Proof by Induction, and the handouts on structural recursion for natural numbers (using the Peano axioms) and Flat lists (as in Racket).

- Proof by induction:
  - Base case: verify the statement is true for the basis of the definition (like $n = 0$ for the natural numbers)
  - Induction case: Assume the statement is true for an arbitrary value (induction hypothesis) like $n$ and use that to show the statement is also true for the next ordered step (induction goal) like $n + 1$.

- The Peano axioms give a recursive definition of the set of natural numbers ($\mathbb{N}$) that gives us the means to write inductive proofs about them. We can similarly define other sets, e.g., the set of lists or the set of all propositional logic statements.
**Exercise: Structural induction**

We can give a recursive definition of what it means to be an even number. Some number \( e \) is even if and only if \( e \) is

<table>
<thead>
<tr>
<th>Base case:</th>
<th>( e = 0 ),</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recursive case:</td>
<td>( e = k + 2 ), where ( k ) is an even number, or</td>
</tr>
<tr>
<td></td>
<td>( e = -k ), where ( k ) is an even number</td>
</tr>
</tbody>
</table>

Use structural induction to prove that every even number, as defined above, is an element of \( E = \{2x \mid x \in \mathbb{Z}\} \), where \( \mathbb{Z} \) is the set of integers.

Don't be scared by structural induction; it's just a version of normal induction where you follow the cases of a recursive definition.
Combinatorics & probability

See Chapters 5 and 6. You may wish to go over the solutions to Assignment 8 and Labs 9 and 10. Beyond what’s covered by these practice problems, make sure you’re familiar with computing an expected value, as covered in class (April 27).

- **Subtraction principle**: $|A - B| = |A| - |A \cap B|$

- **Addition principle**:
  - if $A$ and $B$ are disjoint, $|A \cup B| = |A| + |B|$
  - if $A$ and $B$ are not disjoint, $|A \cup B| = |A| + |B| - |A \cap B|$

- **Multiplication principle**: $|A \times B| = |A| \cdot |B|$

- We count ways of selections of $k$ items from an $n$-element set $A$:
  - **Permutation**: Order matters, repetition not allowed, written $P(n, k) = \frac{n!}{(n-k)!}$
  - **Combination**: Order doesn’t matter, repetition not allowed, written $C(n, k) = \frac{n!}{k!(n-k)!}$. Also denoted \( \binom{n}{k} \). Read as “$n$ choose $k$”.

- Calculating probabilities:
  - Identify the *sample space* of the problem – the non-empty set of possibilities
  - Specify the values of the *probability distribution*, a function on the sample space $S$:
    $$p : S \rightarrow [0, 1] \text{ such that } \sum_{s \in S} p(s) = 1.$$  
    Is it uniform, i.e., every element $s \in S$ has probability $p(s) = 1/|S|$?
  - Identify the *event* of the problem, as a non-empty subset of the sample space $S$ (i.e., a non-empty element of $\mathcal{P}(S)$).
  - Calculate the probability of the event.
    * When the distribution is uniform: $|E|/|S|$
    * When the distribution is not uniform: $\sum p(e)$ for all $e \in E$

- **Probability function**: a function on the power set of a given sample space $S$, using probability distribution $p$:
  $$p^+ : \mathcal{P}(S) \rightarrow [0,1]$$
  - When $A = \emptyset$: $p^+(A) = 0$
  - When $A \subseteq S$ and $A \neq \emptyset$: $p^+(A) = \sum p(s)$ for all $s \in A$.
  - Properties of the probability function:
\[ p(\emptyset) = 0 \]
\[ p(S) = 1 \]

- Unions and probability:
  - When sets \( A \) and \( B \) are disjoint: \( p(A \cup B) = p(A) + p(B) \)
  - When sets \( A \) and \( B \) are not disjoint: \( p(A \cup B) = p(A) + p(B) - p(A \cap B) \)

- Intersections and probability: \( p(A \cap B) = p(A) + p(B) - p(A \cup B) \)

- Combinatorics and probability: when the sample space is large, calculate the number of elements in the sample space and events with permutations or combinations.
Exercise: Counting

For this problem, you should expand your answer, but you don’t need to multiply out factorials. Briefly explain your answers.

a. Consider an odometer – a car’s display of how far it’s driven – with five digits. How many distinct readings are possible?  
   *Hint:* Does the choice of a digit affect any of the other choices?

b. If a restaurant offers four toppings included in the price of every pizza, how many ways could you select four toppings from a full list of eight options for toppings: olives, green peppers, red peppers, onions, eggplant, artichoke, broccoli, and pineapple?  
   *Hint:* The restaurant doesn’t care about the order in which you specify the toppings.

c. How many ways are there to select six volunteers from a group of 25, to fill six different roles?  
   *Hint:* Since each role is unique, the order of the selection matters.
Exercise: Probability

a. A bowl contains 10 balls, numbered from 1 to 10. A ball is randomly selected from the bowl. What is the probability that the number on the ball is an odd integer?

b. Two letters are selected at random from the 25 letters of the English alphabet. What is the probability that both letters selected are one of the five vowels a, e, i, o, or u?