Problem 1

a. If \( f : A \to B \) and \( g : B \to C \) are both injective functions, then their composition, \( g \circ f : A \to C \) must also be injective. However, if \( g \) is not injective, then \( g \circ f \) might not be injective either – even if \( f \) is. Give an example that demonstrates this possibility. (Use a small set \( A \).)

b. Suppose \( f : A \to B \) and \( g : B \to A \) are functions such that for each \( a \in A \), \( g(f(a)) = a \). Circle the statements below that must be true; cross out the statements below that might be false.

- \( f \) is injective
- \( f \) is surjective
- \( g \) is injective
- \( g \) is surjective

c. Write down one of the statements that might be false here: __________________________. Give a counterexample in the space below that demonstrates that it can indeed be false.
**Problem 2**

For this problem, we restrict attention to binary relations over some set $A$ (i.e., relations from $A$ to $A$). We have looked at the properties of a relation being reflexive, symmetric, and transitive:

- A binary relation $R$ over a set $A$ is **reflexive** if and only if for every $x \in A$, $(x, x) \in R$.
- A binary relation $R$ over a set $A$ is **symmetric** if and only if whenever $(x, y) \in R$, then so is $(y, x) \in R$.
- A binary relation $R$ over a set $A$ is **transitive** if and only if whenever $(x, y) \in R$ and $(y, z) \in R$, then so is $(x, z) \in R$.

Here’s a new property:

A binary relation $R$ over a set $A$ is **serial** if and only if for every $x \in A$, there is some $y \in A$ such that $(x, y) \in R$.

For example, the relation, $\{(1, 2)\}$, over the set $A = \{1, 2, 3\}$ is not serial because (for example) $2 \in A$, but there is no $y \in A$ such that $(2, y)$ is in the relation. On the other hand, the relation

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

over the set $A = \{1, 2, 3\}$ is serial since for each $x \in A$ there is at least one $y \in A$ such that $(x, y)$ is in the relation. In this case, each $x \in A$ has three pairs of the form $(x, y)$ in the relation.

a. Give an example of a relation $R$ over the set $\{1, 2, 3\}$ that is serial, but not reflexive.

b. **True or False** (Circle one). If $f : A \rightarrow A$ is a function, then it must be a serial relation.
c. Prove that if a binary relation $R$ over some set $A$ is reflexive, then it must also be serial.

d. Prove that if a binary relation $R$ over some set $A$ is serial, symmetric, and transitive, then it must also be reflexive.
Problem 3

Define a Racket function, called \texttt{listerino}, that takes two integer inputs, \( n \) and \( \text{total} \). It should return as its output a list of fractions, as illustrated below:

\[
\text{> (listerino 4 5)} \\
\text{(1/4 2/3 3/2 4)} \\
\text{> (listerino 6 7)} \\
\text{(1/6 2/5 3/4 4/3 5/2 6)}
\]

Notice that the value of \( n \) is the value of the first denominator in the output list, and that subsequent denominators decrease by 1 each time. Notice that \( \text{total} \) is the sum of the numerator and denominator of any fraction in the output list.

- It may help to write down the recursive function call. Assuming it works properly, what should the recursive function call return as its output?
- The value of \( \text{total} \) should stay the same across the recursive function calls.
**Problem 4**

Claim: $5^n - 2^n$ is divisible by 3 for all $n \in \mathbb{N}$. Prove this using induction.

- For full credit, the logic of your proof must be clear, with each step justified.
- Clearly state the proposition, $P$, that you are using in your proof.

**Hints:**

- A natural number $X$ is divisible by 3 if and only if $X = 3W$ for some natural number $W$.
- You can use standard properties (e.g., $5^{n+1} = 5 \cdot 5^n$ and $5^0 = 1$).
- In the recursive case, you may wish to add the following amount: $5 \cdot 2^n - 5 \cdot 2^n$ (which is, of course, zero).