Russell’s Paradox

In 1902, in a letter to Gottlob Frege, Bertrand Russell drew attention to the danger of unrestricted use of abstraction when forming sets.

Suppose that there are two kinds of sets,

- **Normal sets** – sets that do not contain themselves as elements; and
- **Non-normal sets** – sets that contain themselves as elements.

“The set of all dogs” is a normal set, for obviously the set itself is not a dog.

“The set of all sets” and “the set of all things that are not dogs” are non-normal sets, for the sets are elements of themselves.

Now consider the set $S$ whose elements $x$ are sets that are not elements of themselves,

$$S = \{x \mid x \notin x\}.$$

Is $S$ an element of $S$?

If $S \notin S$, then $S$ meets the requirement $(x \notin x)$ to be a member of $S$ and, paradoxically, $S \in S$.

If $S \in S$, then $S$ fails to meet the requirement $(x \notin x)$ to be a member of $S$ and, paradoxically, $S \notin S$.

The contradiction,

$$\text{if } S \notin S \text{ then } S \in S; \quad \text{if } S \in S \text{ then } S \notin S,$$

is known as **Russell’s paradox**.

We may see the above as a proof of the theorem that

$$S = \{x \mid x \notin x\} \text{ does not exist.}$$
To enact his discovery, Russell proposed, in 1918, the **barber paradox**.

A barber has a sign:

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I shave all those men in
town, and only those men,
who do not shave themselves.
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A non-member of the set of all men in town who do not shave themselves – a *woman* – can, of course, advertise as above with impunity.

If a man, and he shaved *himself*, the barber would belong to the set of men who shave themselves, but the sign implies that he shaves *only* those who do not shave themselves, so he cannot shave himself – and neither can anybody else, as he, the barber, shaves *all* those men who do not shave themselves. If he decided to wear a full beard, the contradiction prevails, for the sign says that *he* shaves *all* those men who do not shave themselves.

Russell’s paradox was not the first paradox to be discovered in set theory, but its simplicity and directness had an immense impact on the development of the ideas and foundations of mathematics at the beginning of the 20th century; before that time the concept “class of all classes” or “set of all sets” had been used in an unhampered manner.