Definition. A relation, $R$, on the natural numbers is called *orderly* if it satisfies the following two conditions:

1. $(0, S(n)) \in R$, for all $n \in \mathbb{N}$; and
2. for any $k, n \in \mathbb{N}$, if $(k, n) \in R$, then $(S(k), S(n)) \in R$.

Theorem 1. If $R$ and $S$ are both orderly relations, then so is their intersection, $R \cap S$.

Proof 1. Since $R$ and $S$ are both orderly, they both satisfy condition (1). Thus, for any $n \in \mathbb{N}$, we have that $(0, S(n)) \in R$ and $(0, S(n)) \in S$. Therefore, $(0, S(n)) \in R \cap S$. Thus, $R \cap S$ also satisfies condition (1).

Next, suppose $(k, n) \in R \cap S$. Then $(k, n) \in R$ and $(k, n) \in S$. Since $R$ and $S$ are both orderly, they both satisfy condition (2). Thus, $(S(k), S(n)) \in R$ and $(S(k), S(n)) \in S$. Therefore, $(S(k), S(n)) \in R \cap S$. Thus, $R \cap S$ also satisfies condition (2).

Definition. Let $R_<$ be the intersection of all the orderly relations over the natural numbers. Stated differently, $R_<$ is the smallest relation on the natural numbers such that:

1. $(0, S(n)) \in R_<$, for all $n \in \mathbb{N}$; and
2. for any $k, n \in \mathbb{N}$, if $(k, n) \in R_<$, then $(S(k), S(n)) \in R_<$.

In other words, $R_<$ is the smallest orderly relation on the natural numbers.

We can, of course, use a more familiar notation for this relation, as follows.

Equivalent Definition. Let $<$ be the smallest relation on the natural numbers such that:

1. $0 < S(n)$ for all $n \in \mathbb{N}$; and
2. for any $k, n \in \mathbb{N}$, if $k < n$, then $S(k) < S(n)$.

Theorem 2. For every $m \in \mathbb{N}$, it is not the case that $m < 0$. In other words: $m \not< 0$.

1 Handout written by Luke Hunsberger, 2010
2 The term *orderly* is not standard; I made it up for this handout.
Proof 2. Suppose $m < 0$ for some $m \in \mathcal{N}$. In other words, $(m, 0) \in R_\prec$. But in that case we could define a new relation, $R' = R_\prec - \{(m, 0)\}$. In other words, $R'$ consists of all pairs in $R_\prec$ except $(m, 0)$. Next, we show that $R'$ is orderly, as follows.

First, by condition (1) of being orderly, we know that $(0, S(n)) \in R_\prec$ for all $n \in \mathcal{N}$. Furthermore, by one of Peano’s axioms we know that $S(n) \neq 0$. Thus, the pair $(0, S(n))$ is not the same as the pair $(m, 0)$. Thus, the pair $(0, S(n))$ was not removed from $R'$. Thus, $(0, S(n)) \in R'$ for all $n \in \mathcal{N}$. In other words, removing $(m, 0)$ from $R'$ did not cause $R'$ to violate condition (1).

Next, suppose $(k, n) \in R' \subset R_\prec$. Since $R_\prec$ satisfies condition (2), it must be that $(S(k), S(n)) \in R_\prec$. However, since $0 \neq S(n)$ for any $n$ (Peano again), it must be that $(S(k), S(n)) \in R'$. In other words, removing $(m, 0)$ from $R'$ did not cause $R'$ to violate condition (2).

Thus, $R'$ is a proper subset of $R_\prec$ that is orderly. Oops! That contradicts the definition of $R_\prec$! Thus, our assumption that $m < 0$ must have been wrong.

Notice that condition (2) of being orderly goes in only one direction. The following theorem gives us the other direction – but only for the relation $R_\prec$. This other direction does not hold for all orderly relations.

Theorem 3. For any $k, n \in \mathcal{N}$, if $S(k) < S(n)$, then $k < n$.

Proof 3. Suppose $S(k) < S(n)$, but $k \nleq n$, for some $k, n \in \mathcal{N}$. Then we can define a new relation, $R' = R_\prec - \{(S(k), S(n))\}$. In other words, $R'$ is the same as $R_\prec$, except that it doesn’t contain the pair $(S(k), S(n))$. We can show that $R'$ is orderly, as follows.

First, for any $n$, $(0, S(n)) \in R_\prec$. And since that pair was not removed from $R'$ (since $S(n) \neq 0$ for any $n$, courtesy of Peano), we must have $(0, S(n)) \in R'$. Thus, $R'$ satisfies condition (1).

Next, suppose $(a, b) \in R'$. Since $R' \subseteq R_\prec$, it must be that $(a, b) \in R_\prec$. Since $R_\prec$ satisfies condition (2), it must be that $(S(a), S(b)) \in R_\prec$. Furthermore, since $(a, b) \in R_\prec$ but $(k, n) \not\in R_\prec$, it must be that $(a, b)$ and $(k, n)$ are distinct pairs. In other words, either $a \neq k$ or $b \neq n$. However, Peano tells us that different numbers have different successors; thus, we get in turn that either $S(a) \neq S(k)$ or $S(b) \neq S(n)$. And therefore, $(S(a), S(b))$ and $(S(k), S(n))$ are distinct pairs. Since $(S(k), S(n))$ was the only pair removed from $R'$, it must be that $(S(a), S(b)) \in R'$. Thus, $R'$ satisfies condition (2).

However, $R'$ being an orderly proper subset of $R_\prec$ contradicts the definition of $R_\prec$. Thus, our assumption that $k \nleq n$ must have been wrong.

In other words, it is not possible to have $S(k) < S(n)$ without having $k < n$, too.

Corollary to Theorem 3. For any $k, n \in \mathcal{N}$, $k < n$ if and only if $S(k) < S(n)$.

Theorem 4. For any $k \in \mathcal{N}$, it must be that $k < S(k)$.

Proof 4. Let $P(n)$ be the proposition: $n < S(n)$.

- **Base Case:** $P(0)$ is the proposition: $0 < S(0)$. This holds since $0 < S(n)$ for any $n \in \mathcal{N}$ (by condition (1) of being orderly).

- **Recursive Case:** $P(S(k))$ is the proposition: $S(k) < S(S(k))$. By the corollary to Theorem 3, this holds if and only if $k < S(k)$, which holds by the inductive hypothesis, $P(k)$. 

Theorem 5. For every \( a \in \mathbb{N} \), \( a \not< a \).

Proof 5. Let \( P(n) \) be the proposition that \( n \not< n \).

- **Base Case:** \( P(0) \) is the proposition that \( 0 \not< 0 \), which holds by Theorem 2.

- **Recursive Case:** \( P(S(k)) \) is the proposition that \( S(k) \not< S(k) \). By the corollary to Theorem 3, this holds if and only if \( k \not< k \), which holds by the inductive hypothesis, \( P(k) \).

Theorem 6. For all \( a, b \in \mathbb{N} \), either \( a \not< n \) or \( n \not< a \).

Proof 6. Let \( P(n) \) be the proposition: for all \( a \in \mathbb{N} \), either \( a \not< n \) or \( n \not< a \).

- **Base Case:** \( P(0) \) is the proposition: for all \( a \in \mathbb{N} \), either \( a \not< 0 \) or \( 0 \not< a \). By Theorem 2, we have that \( a \not< 0 \) for all \( a \). Thus, \( P(0) \) holds.

- **Recursive Case:** \( P(S(k)) \) is the proposition: for all \( a \in \mathbb{N} \), either \( a \not< S(k) \) or \( S(k) \not< a \). There are two cases to consider:
  (i) If \( a = 0 \), then \( S(k) \not< a \) by Theorem 2.
  (ii) If \( a = S(w) \) for some \( w \in \mathbb{N} \), then \( a \not< S(k) \) or \( S(k) \not< a \) holds if and only if \((S(w) \not< S(k) \) or \( S(k) \not< S(w)) \) (by direct substitution). But by the corollary to Theorem 3, this holds if and only if \((w \not< k \) or \( k \not< w)) \), which holds by the inductive hypothesis, \( P(k) \).

Thus, \( P(S(k)) \) holds if \( P(k) \) holds.

Theorem 7. For all \( a, b \in \mathbb{N} \), if \( a \not< b \) and \( a \neq b \), then \( b < a \). In other words, if \( a \neq b \) then either \( a < b \) or \( b < a \).

Proof 7. Let \( P(n) \) be the proposition: for all \( b \in \mathbb{N} \), if \( n \not< b \) and \( n \neq b \), then \( b < n \).

- **Base Case:** \( P(0) \) is the proposition: for all \( b \in \mathbb{N} \), if \( 0 \not< b \) and \( 0 \neq b \), then \( b < 0 \). Well, if \( 0 \neq b \), then \( b = S(w) \) for some \( w \in \mathbb{N} \), in which case \( 0 < b \) by condition (1) of being orderly. Since the premise of the implication is always false, the implication itself is always true.

- **Recursive Case:** \( P(S(k)) \) is the proposition: for all \( b \in \mathbb{N} \), if \( S(k) \not< b \) and \( S(k) \neq b \), then \( b < S(k) \). There are two cases to consider:
  (b = 0) In this case, \( b < S(k) \) holds by condition (1) of being orderly.
  (b = S(w) for some \( w \in \mathbb{N} \)) In this case, \( S(k) \not< b \) iff \( S(k) \not< S(w) \) iff \( k \not< w \) (by corollary to Theorem 3). Similarly, \( S(k) \neq b \) iff \( S(k) \neq S(w) \) iff \( k \neq w \), since different numbers have different successors (by Peano). Thus, if \( S(k) \not< b \) and \( S(k) \neq b \), then \( k \not< w \) and \( k \neq w \) and, hence, by the inductive hypothesis (i.e., \( P(k) \)), \( w < k \); therefore, \( S(w) < S(k) \) (by Condition (2)); in other words: \( b < S(k) \).

Thus, \( P(S(k)) \) holds whenever \( P(k) \) holds.
**Theorem 8.** The $<$ relation is transitive.

**Proof 8.** Let $P(n)$ be the proposition: for all $b, c \in \mathcal{N}$, if $n < b$ and $b < c$, then $n < c$.

- **Base Case:** $P(0)$ is the proposition: for all $b, c \in \mathcal{N}$, if $0 < b$ and $b < c$, then $0 < c$. First, $b < c$ implies that $c \neq 0$ (by Theorem 2). But then $0 < c$ follows from condition (1) of being orderly.

- **Recursive Case:** $P(S(k))$ is the proposition: for all $b, c \in \mathcal{N}$, if $S(k) < b$ and $b < c$, then $S(k) < c$. First, $S(k) < b$ implies $b \neq 0$ (by Theorem 2). Similarly, $b < c$ implies that $c \neq 0$. But then $b = S(w)$ for some $w \in \mathcal{N}$; and $c = S(v)$ for some $v \in \mathcal{N}$. Therefore, we have $S(k) < S(w)$, and thus $k < w$ (by corollary to Theorem 3); similarly, we have $S(w) < S(v)$, which implies that $w < v$. Next, since $k < w$ and $w < v$, the inductive hypothesis $P(k)$ ensures that $k < v$. But then $S(k) < S(v)$ (i.e., $S(k) < c$), as desired.

Now that we know a lot about the $<$ relation on $\mathcal{N}$, we can define the $\leq$ relation on $\mathcal{N}$.

**Definition.** Let $\leq$ be the relation on $\mathcal{N}$ defined by: $a \leq b$ if and only if $a = b$ or $a < b$.

**Theorem 9.** The $\leq$ relation is reflexive, transitive and anti-symmetric.

**Proof 9.**

- **Reflexivity:** Let $n$ be some arbitrary element of $\mathcal{N}$. Since $n = n$, the definition of $\leq$ stipulates that $n \leq n$.

- **Transitivity:** Suppose $a, b, c \in \mathcal{N}$ such that $a \leq b$ and $b \leq c$. There are several cases to consider:
  - $(a = b)$ In this case, $b \leq c$ implies that $a \leq c$.
  - $(b = c)$ In this case, $a \leq b$ implies that $a \leq c$.
  - $(a \neq b$ and $b \neq c)$ In this case, $a < b$ and $b < c$. But then $a < c$ follows from the transitivity of $<$. And so $a \leq c$.

- **Anti-Symmetry:** Suppose $a, b \in \mathcal{N}$ such that $a \leq b$ and $a \neq b$. We must show that $b \leq a$. Well, $a \neq b$ implies that $a \neq b$ and $a \neq b$. In that case, Theorem 7 tells us that $b < a$, and hence $b \leq a$.

**Theorem 10.** For any $k, m \in \mathcal{N}$, $k < S(m)$ if and only if $k \leq m$.

**Proof 10.** Let $P(n)$ be the proposition: for all $m \in \mathcal{N}$, $n < S(m)$ if and only if $n \leq m$.

- **Base Case:** $P(0)$ is the proposition: for all $m \in \mathcal{N}$, $0 < S(m)$ if and only if $0 \leq m$. First, $0 < S(m)$ holds for all $m$ by condition (1) of being orderly. Second, if $m = 0$ then $0 \leq m$ holds (by definition). But if $m \neq 0$, then $m = S(w)$ for some $w$ and we have that $0 < S(w) = m$ by condition (1). Thus, $0 \leq m$ for all $m \in \mathcal{N}$.
• **Recursive Case:** $P(S(k))$ is the proposition: for all $m \in \mathcal{N}$, $S(k) < S(m)$ if and only if $S(k) \leq m$.

$(\Rightarrow)$ Suppose $S(k) < S(m)$. Then, $k < m$ by Theorem 3. Thus, $m \neq 0$ by Theorem 2. Thus, $m = S(z)$ for some $z \in \mathcal{N}$, and $k < S(z)$. But then the inductive hypothesis, $P(k)$, ensures that $k \leq z$ (i.e., $k = z$ or $k < z$). If $k = z$, then $S(k) = S(z)$; if $k < z$, then $S(k) < S(z)$ by condition (2) of being orderly. Thus, $S(k) \leq S(z) = m$.

$(\Leftarrow)$ Suppose $S(k) \leq m$.

- Case 1: $S(k) = m$. Now $m < S(m)$ by Theorem 4. So, $S(k) = m < S(m)$.
- Case 2: $S(k) < m$. But $m < S(m)$, by Theorem 4. Thus, $S(k) < m < S(m)$. Since $<$ is transitive (Theorem 8), we have that $S(k) < S(m)$.

The following theorem provides a variant of induction, known as **Strong Induction.** It can be quite useful in certain circumstances, as we shall see! Notice that the statement of strong induction makes use of the $<$ relation.

**Theorem 11: Strong Induction.** Let $P$ be some proposition on natural numbers. Suppose that whenever $P(k)$ holds for all $k < n$, $P(n)$ holds. (We’ll call this the strong induction (SI) property.) Then $P(n)$ holds for all $n$.

**Proof 11.** Suppose $P$ is some proposition that satisfies the SI property. Let $Q(w)$ be the related proposition: $P(k)$ holds for all $k \leq w$. We shall use ordinary induction to show that $Q(w)$ holds for all $w \in \mathcal{N}$.

- **Base Case:** $Q(0)$ is the proposition that $P(k)$ holds for all $k \leq 0$. Since 0 is the only natural number that is less than or equal to 0, $Q(0)$ holds if and only if $P(0)$ holds. The SI property (for $n = 0$) says that if $P(k)$ holds for all $k < 0$, then $P(0)$ holds. Since there are no $k < 0$, the premise is trivially true; hence $P(0)$ holds.

- **Recursive Case:** $Q(w)$ is the proposition that $P(k)$ holds for all $k \leq w$. By Theorem 10, $k \leq w$ if and only if $k < S(w)$. Thus, $Q(w)$ is equivalent to: $P(k)$ holds for all $k < S(w)$. But then the SI property for $n = S(w)$ implies that $P(S(w))$ holds. But then $Q(w)$ and $P(S(w))$ together imply that $P(k)$ holds for all $k \leq S(w)$—i.e., that $Q(S(w))$ holds. Thus, $Q(S(w))$ holds whenever $Q(w)$ holds.

By ordinary induction, $Q(w)$ holds for all $w \in \mathcal{N}$. But this implies that $P(w)$ holds for all $w$. Thus, if $P$ satisfies the SI property, then $P(w)$ holds for all $w \in \mathcal{N}$.

**Theorem 12.** Every non-empty subset of $\mathcal{N}$ has a least element. In other words, for every $A$ such that $\emptyset \neq A \subseteq \mathcal{N}$, there is some $a \in A$ such that $a \leq b$ for all $b \in A$.

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3 This way of formulating and proving Theorems 11 and 12 draws heavily from Alley Stoughton’s web page at [http://people.cis.ksu.edu/~stough/forlan](http://people.cis.ksu.edu/~stough/forlan).
Suppose $A$ is a subset of $\mathbb{N}$ that does not have a least element. We’ll use strong induction to prove that $A$ must be empty, as follows: Let $P(n)$ be the proposition that $n \notin A$. Let $n$ be some arbitrary element of $\mathbb{N}$ such that $P(k)$ holds for all $k < n$ (i.e., $k \notin A$ for all $k < n$). If $P(n)$ does not hold, then $n \in A$. But then $n$ would be the least element of $A$ (since $k < n$ implies that $k \notin A$). Thus, since $A$ does not have a least element, $P(n)$ must hold. But then we have shown that the Strong Induction (SI) property holds for $P$. Thus, by strong induction, $P(n)$ holds for all $n$. In other words, $n \notin A$ for all $n$. Thus, $A = \emptyset$. ■