Equivalence relations and partitions

Assignment 1 solutions are out.
Assignment 2 due on Wednesday.

Previously:
Relations and their properties

Today:
Equivalence relations
Partitions
And their connection

Review: Properties of relations

Binary relations

DEFINITION. For any sets A and B, a binary relation from A to B is a subset of the Cartesian product A×B.

A relation is determined by the ordered pairs it covers.

\[ \text{Has-as-pet} = \{ (\text{John, dog}), \quad (\text{John, rabbit}), \quad (\text{Mary, cat}), \quad (\text{Mary, rabbit}) \} \]
Transitivity, reflexivity, symmetry

A binary relation $R$ is **transitive** iff whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

A binary relation $R$ is **reflexive** over a set $A$ iff for all $a \in A$, $(a, a) \in R$.

A binary relation $R$ is **symmetric** iff whenever $(a, b) \in R$, $(b, a) \in R$.

Equivalence relations

If a relation is reflexive and symmetric, it’s a **similarity relation**.

If a relation is reflexive, symmetric, and transitive, it’s an **equivalence relation**.

Equivalence relations

If $R$ is an equivalence relation, then it induces **equivalence classes** on the elements.

For equivalence relation $R$ and any element $a$, let $C_a$ stand for all elements related to element $a$ in $R$ – that is, $C_a = \{b \mid (a, b) \in R\}$.

Then $C_a = C_b$ exactly when $(a, b) \in R$. 
Examples of equivalence relations

What are some equivalence relations over the natural numbers?

- Numbers $a$ and $b$ are both even
- Numbers $a$ and $b$ are both odd

- Polygons $a$ and $b$ have the same number of sides
- Persons $a$ and $b$ have the same nationality (?)

Equivalence vs identity

Equivalence relations may be written as $a \approx b$ rather than $(a, b) \in R$ to emphasize that they behave like identity.

Identity (=) allows substitution in all cases; equivalence ($\approx$) does not.

Partitions

Intuitively, what does it mean to you if something – or a collection of things – is partitioned?

How could we write that definition in rigorous, formal notation?
Partitions

When we *partition* a set, we divide it into categories such that every element belongs to exactly one category.

DEFINITION. A *partition* of a non-empty set $A$ is defined to be any collection $\{B_i\}_{i \in I}$ of non-empty *subsets* of $A$ such that $\{B_i\}_{i \in I}$ is called an *index set* mutually *exhausts* $A$: $\cup \{B_i\}_{i \in I} = A$, that is, iff every element of $A$ is in at least one of the subsets $B_i$.

is *pairwise disjoint*: every two distinct sets $B_i$ in the collection are disjoint, i.e., for all $i, j \in I$, if $B_i \neq B_j$, then $B_i \cap B_j = \emptyset$.

The sets $B_i$ are called the *cells* or *blocks* of the partition.

Exercise

$A = \{1, 2, 3, 4\}$

Which of the following are partitions of $A$?

- $\{\{1, 2\}, \{3\}\}$: NO: does not cover 4
- $\{\{1, 2\}, \{2, 3\}, \{4\}\}$: NO: overlap – not pairwise disjoint
- $\{\{1, 2\}, \{3, 4\}, \emptyset\}$: NO: definition says non-empty subsets
- $\{\{1, 4\}, \{3, 2\}\}$: YES
Exercise

We say that one partition of a set is at least as fine as another iff every cell of the former is a subset of a cell of the latter.

$A = \{1, 2, 3, 4\}$

What’s the finest partition?

$\{\{1\}, \{2\}, \{3\}, \{4\}\}$

The least fine?

$\{\{1, 2, 3, 4\}\}$

Connecting equivalence relations and partitions

Equivalence relations and partitions seem like different ways of expressing the same thing.

It looks like every equivalence relation over $A$ determines a partition over $A$.

It looks like every partition over $A$ determines an equivalence relation over $A$.

Let’s prove it!

Theorem 1

Let $R$ be an equivalence relation over the set $A = \{a_1, a_2, \ldots, a_n\}$.

Let $P = \{B_1, B_2, \ldots, B_n\}$ be a collection of subsets of $A$ where for each $i$, $B_i = \{a \mid (a, a) \in R\}$. This is called the image of $a$ under $R$.

For example, $B_1$ is the set of $a$s that are related to $a_1$; $B_2$ is the set of $a$s that are related to $a_2$; and so on.

Then $P$ is a partition of $A$. 
Sidebar: Image

Let $R$ be a relation from set $A$ to set $B$.
Let $a \in A$.

The image of $a$ under $R$, written $R(a)$, is the set of all $b \in B$ such that $(a, b) \in R$.

So, if the relation $R$ is $<$, then the image $<(10)$ would be every number less than 10.

Proof of Theorem 1

Let $R$ be an equivalence relation over set $A$, and $P$ be a set of the image of each element of $A$ under $R$, as defined in the theorem.

Goal: Show that $P$ is a partition over $A$.

Thus, we must show that:

1. Each $B_i$ is a non-empty subset of $A$.
2. The collection $\{B_1, B_2, ..., B_n\}$ exhausts $A$.
3. The elements of $P$ are pairwise disjoint.

Proof of Goal 1.

Show that each $B_i$ is a non-empty subset of $A$.

Since $R$ is an equivalence relation, it is reflexive. Thus, for each $a_i \in A$, $(a_i, a_i) \in R$.

In that case, each set $B_i$ contains at least one element, namely, $a_i$. Thus, each $B_i$ is non-empty.

Furthermore, each $B_i \subseteq A$ since $R$ is a relation over $A$; thus, the only things that can be related to $a_i$ are elements of $A$.

Proof of Goal 2.

Show that the collection $\{B_1, B_2, ..., B_n\}$ exhausts $A$.

Let $U$ be the union of all the $B_i$s.

We seek to show that $A = U$, i.e.,

$U \subseteq A$

$A \subseteq U$
Proof of Goal 2.

Show that the collection \{B_1, B_2, ..., B_n\} exhausts A.
Let U be the union of all the B_i's.
We seek to show that A = U, i.e.,
\[ U \subseteq A \]
Since each B_i \subseteq A, it follows that U \subseteq A.
Suppose x is any element of U.
Thus, x is an element of at least one of the B_i subsets.
Thus there is some q \in A, such that x \in B_q.
(\text{Remember, each one of the B_i subsets derives from a particular element of A.})
Now, x being an element of B_q implies that (q, x) \in R.
Then x must be an element of A.
(\text{Since R is a relation over A.})
A \subseteq U

Proof of Goal 2, continued.

Show that the collection \{B_1, B_2, ..., B_n\} exhausts A.
Let U be the union of all the B_i's.
We seek to show that A = U, i.e.,
\[ U \subseteq A \]
Since each B_i \subseteq A, it follows that U \subseteq A.
\[ A \subseteq U \]
In other words, we need to show that for any \( x \), if \( x \in A \), then \( x \in U \).
Since R is reflexive, \( (x, x) \in R \).
Thus \( x \in B \) and \( x \in U \).

Proof of Goal 3.

The elements of P are pairwise disjoint.
We seek to show that for any sets B_i and B_j in P, if B_i \cap B_j is non-empty, then B_i = B_j.
Let B_i and B_j be any sets in P such that B_i \cap B_j is non-empty.
We must show that B_i = B_j (i.e., B_i \subseteq B_j and B_j \subseteq B_i).

Proof of Goal 3, continued.

We seek to show that for any sets B_i and B_j in P, if B_i \cap B_j is non-empty, then B_i = B_j...
Show B_i \subseteq B_j.
Let x be any element of B_i. We must show that x \in B_j.
By assumption, B_i \cap B_j is non-empty. Thus, there is some y in both B_i and B_j. By definition of the sets B_i and B_j, it follows that \( (a, y) \) and \( (a, y) \) are in R.
Since \( (a, x) \in R \) and R is symmetric, it follows that \( (a, x) \in R \).
Thus, \( (x, a) \) and \( (a, y) \) are both in R. Since R is transitive, it follows that \( (x, y) \in R \).
Since \( (a, y) \in R \) and R is symmetric, it follows that \( (a, x) \in R \).
Thus, both \( (x, y) \) and \( (y, a) \) are in R. Since R is transitive, it follows that \( (x, a) \in R \).
And by symmetry, \( (a, x) \) must be in R.
But then, by definition of B_j, it follows that x \in B_j.
Proof of Goal 3, continued.

We seek to show that for any sets $B_i$ and $B_j$ in $P$, if $B_i \cap B_j$ is non-empty, then $B_i = B_j$...

✓ Show $B_i \subseteq B_j$

Show $B_j \subseteq B_i$.

By a symmetric argument, you can show $B_j$ is also a subset of $B_i$.

That concludes the proof that there is a corresponding partition for every equivalence relation.

Notice that we used all three properties of an equivalence relation.

If $R$ is not an equivalence relation, then the proof will not work.

Next

Show the other direction: There’s a corresponding equivalence relation for every partition.