Assignments:
- Assignment 2 due now.
- Assignment 3 out later today.

Previously:
- Relations and their properties
- Showed there’s a partition for every equivalence relation

Today:
- Show there’s an equivalence relation for every partition
- Partial, total, and strict orderings
- Closures

Equivalence relations

*R* is an **equivalence relation** over a set *A* iff it is

- **reflexive**: for all *a* ∈ *A*, (*a, a*) ∈ *R*.
- **symmetric**: whenever (*a, b*) ∈ *R*, (*b, a*) ∈ *R*.
- **transitive**: whenever (*a, b*) ∈ *R* and (*b, c*) ∈ *R*, then (*a, c*) ∈ *R.*
Partitions

\{B_i\}_{i\in I} is a partition of a non-empty set \(A\) iff

- every \(B_i\) is a non-empty subset of \(A\)
- \(\{B_i\}_{i\in I}\) mutually exhausts \(A\): \(\cup\{B_i\}_{i\in I} = A\). That is, every element of \(A\) is in at least one of the subsets \(B_i\).

is pairwise disjoint: for all \(i, j \in I\), if \(B_i \neq B_j\), then \(B_i \cap B_j = \emptyset\). That is, every two distinct sets in the partition are disjoint.

Equivalence relations and partitions seem like different ways of expressing the same thing.

Prove:

✓ Every equivalence relation over \(A\) determines a partition over \(A\).

Every partition over \(A\) determines an equivalence relation over \(A\).

Theorem 2

Let \(A\) be any non-empty set.

Let \(P = \{B_1, B_2, \ldots, B_m\}\) be any partition of \(A\).

Let \(R\) be the following relation:

\[ R = \{(a, b) \mid \text{there is a set } B_i \text{ such that } a \text{ and } b \text{ both belong to } B_i\} \]

Note: We don’t assume that there is only one such \(B_i\). (But it will turn out to be true.)

Then \(R\) is an equivalence relation.

Proof of Theorem 2

Let \(P, A,\) and \(R\) be as in the statement of the theorem.

We seek to show that \(R\) is an equivalence relation, i.e.,

- \(R\) is reflexive.
- \(R\) is symmetric.
- \(R\) is transitive.
Proof of Theorem 2

We seek to show that $R$ is an equivalence relation, i.e.,

- $R$ is reflexive, i.e., for all $a$ in $A$, $(a, a) \in R$.
  - Let $x$ be any element of $A$.
  - Since $P$ is a partition of $A$, the sets $K_i$ exhaust $A$. Thus, $A = \cup B_i$. Thus, $x \in \cup B_i$.
  - That implies that $x$ must be an element of at least one of the $B_i$ (by definition of a union).
  - Thus, there is some $B_i$ such that both $x$ and $x$ belong to $B_i$. Thus, $(x, x) \in R$. (Recall the definition of $R$.)
- $R$ is symmetric.
- $R$ is transitive.
Equivalence relations and partitions seem like different ways of expressing the same thing.

Prove:

✓ Every equivalence relation over $A$ determines a partition over $A$.
✓ Every partition over $A$ determines an equivalence relation over $A$.

Why not empty partitions?

To prove Theorem 2, we used the requirements that the sets of a partition be pairwise disjoint and that they exhaust $A$.

However, we did not use the fact that the sets in a partition must be non-empty.

Consider this example:

$A = \{1, 2\}$
$P = \{\{1, 2\}, \emptyset\}$

The sets in $P$ satisfy all the requirements of a partition except that one of the sets is empty.

Nonetheless, we can define a relation $R$ by:

$R = \{(a, b) | \text{there is some set in } P \text{ that contains both } a \text{ and } b\}$

In this case, we get $R = \{(1, 1), (1, 2), (2, 2), (2, 1)\}$, which is indeed an equivalence relation.

This example does not invalidate Theorem 1 or Theorem 2 in any way.

It just shows that the requirement that the sets in a partition be non-empty is not really relevant for those theorems.

Nonetheless, there's a good reason to exclude empty sets from partitions.

The following point from Charles Wells, 2009,
http://www.abstractmath.org/MM/MMEquivalenceRelations.htm
Suppose $A = \{1, 2, 3\}$

$A$ can be partitioned into non-empty subsets five ways:

$P_1 = \{\{1\}, \{2\}, \{3\}\}$
$P_2 = \{\{1\}, \{2, 3\}\}$
$P_3 = \{\{1, 2\}, \{3\}\}$
$P_4 = \{\{1, 3\}, \{2\}\}$
$P_5 = \{\{1, 2, 3\}\}$

There are also five possible equivalence relations over $A$:

$Q_1 = \{(1,1), (2,2), (3,3)\}$
$Q_2 = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$
$Q_3 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$
$Q_4 = \{(1,1), (1,3), (3,1), (3,3), (2,2)\}$
$Q_5 = \{(1,1), (1,2), (1,3), (2,3), (2,1), (3,1), (3,2), (2,2), (3,3)\}$

Notice that the corresponding partitions and equivalence relations are those that are constructed in Theorem 1 and Theorem 2.

Thus, we can define a function, $f$, from partitions to equivalence relations, where: $f(P_i) = Q_i$.

This function is a bijection! Its inverse is the function $g$, where $g(Q_i) = P_i$.

**One more thing**

Proving Theorem 1, we took an equivalence relation and defined a partition from it.

Proving Theorem 2, we took a partition and defined an equivalence relation from it.

So, if take an equivalence relation $R$, turn it into a partition $P$, and then turn $P$ back into a relation $R'$, will $R = R'$?

Find out in an exciting handout!

Relations: *Orderings*
Using relations to order elements

We can use relations to order things, e.g., the numeric less than relation (<):

Should it be transitive?
Yes!
Should it be reflexive?
Yes if we want it to be inclusive, e.g., ≤.
No if we want strict order, e.g., <.

Ordering and symmetry

Given a reflexive, transitive relation that we want to use as an ordering, should it be symmetric?

No! That would make it an equivalence relation.
We want it to be antisymmetric: whenever \((a, b) \in R\) then \((b, a) \notin R\) provided \(a \neq b\).
   Equivalently, whenever \((a, b) \in R\) and \((b, a) \in R\), then \(a = b\).

Partial ordering

If a relation \(R\) over a set \(A\) is reflexive, transitive, and antisymmetric – by the preceding definition – it is a partial order of \(A\).

Examples:
- For sets, the subset relation is a partial order. It’s reflexive and transitive. It’s antisymmetric since whenever \(A \subseteq B\) and \(B \subseteq A\), then \(A = B\).
- In arithmetic, a being a divisor of \(b\) is a partial ordering over the positive integers. \(R\) over \(\mathbb{N}^+\) = \{\((a, b) \mid b = ka\) for some \(k\) in \(\mathbb{N}^+\)\}.

Total ordering

A reflexive relation \(R\) over a set \(A\) is complete over \(A\) iff for all \(a, b \in A\), either \((a, b) \in R\) or \((b, a) \in R\) (or both).

A (relation, set) pair – also called a poset – where the relation is complete over the set is called a total ordering or linear ordering.
Total ordering examples

Total orderings:

\( \leq \) over \( \mathbb{N} \)

Dictionary order of words

Not total orderings:

\( < \) over \( \mathbb{N} \)

\( \subseteq \) over \( \mathcal{P}(A) \). Consider singletons \{a\} and \{b\}. Both are elements of \( \mathcal{P}(A) \), but neither \{a\} \( \subseteq \) \{b\} nor \{b\} \( \subseteq \) \{a\}.

Strict ordering

Whenever we have a reflexive relation like \( \leq \), we can separate its strict part, \( < \):

\[
a < b \text{ iff } a \leq b \text{ but } a \neq b.
\]

\[
(\forall a b. a < b) = (\forall a b. a \leq b \land a \neq b)
\]

Transitive closure

Given a relation \( R \) that isn’t transitive, we can compute the transitive closure of \( R \), the unique, smallest superset of \( R \) that is transitive.

Can be written as \( R^* \).

Why smallest?

Any relation \( R \) over \( A \) is a subset of \( A^2 \), which is transitive, but you’ve also lost the meaning that \( R \) was encoding!
Bottom-up definition of transitive closure

\[
R_0 = R \\
R_{n+1} = R_n \cup \{(a, c) \mid \exists x \text{ s.t. } (a, x) \in R_n \text{ and } (x, c) \in R\} \\
R^* = \bigcup \{R_n \mid n \in \mathbb{N}\}
\]

Computationally, we keep doing this until there are no new pairs \((a, c)\) to add.
If there are always more to add? We go on forever, since the transitive closure is infinite.

Transitive closure example

What \(R_i\) are constructed to find the transitive closure of a relation \(R\) over \(\{1, 2, 3, 4\}\) when

\[R = \{(1, 2), (2, 3), (3, 4)\}\]
\[R = \{(2, 3), (2, 4), (3, 2), (4, 1), (4, 3)\}\]

Closure under a relation

More generally, we can define \(R[A]\), the closure of a set \(A\) under relation \(R\):

\[A_0 = A \]
\[A_{n+1} = A_n \cup R(A_n) \text{ for each natural number } n\]
\[R[A] = \bigcup \{A_n \mid n \in \mathbb{N}\}\]

Recall that \(R(A_n)\) is the image of set \(A_n\) under relation \(R\): \(\{b \mid \exists x \in X \text{ such that } (x, b) \in R\}\).
So, we apply the relation over and over finding the set of all elements that were in the original set and those that can be added to it by applying the relation.