Functions

A function is a rule that associates an input, the argument of the function, with an output, the value of the function.

\[ f(x) = x^2 \]
Some functional history

1692: The term “function” – Latin *funcio* – first appears in an article in the *Acta Eruditorum* to denote various tasks a straight line may accomplish with respect to a curve, e.g., forming a chord, tangent, or normal.

Article is signed “O.V.E.” but believe to be by German mathematician Gottfried von Leibniz.

1694: In another article in *Acta Eruditorum*, Leibniz gives the term a more specific meaning by letting it denote the slope of a curve.

This definition has little in common with the present-day mathematical definition.

1794: Swiss mathematician Leonhard Euler defines a function as a variable quantity that is dependent on another quantity. Closer!

1822: French physicist and mathematician Joseph Fourier presented work on heat flow (*Théorie analytique de la chaleur*). He introduced series with sines and cosines as terms, leading to the concept that a representation of a function may be valid *only for a certain range of values*.

1837: German mathematician Johann Lejeune Dirichlet proposed that a mathematical function is a correspondence that assigns a unique value of the dependent variable to every permitted value of an independent variable.

Traditional functions

The argument and value would be numbers and the function relating them would be an equation.

The argument is an independent variable that causally affects the value.

Traditional functions: Example

If we buy coffee and pay by the mass, we can say that the price is a function of the mass.

If coffee is $9/kg, half a kg is $4.50, 2 kg is $18, etc. That is, we have the formula

\[ f(x) = 9x \]

where \( x \) designates the purchased amount (kg) and \( f(x) \) is the price (in $). Here, \( f(x) \) is defined only for \( x \geq 0 \).
Modern functions

The argument and value an be any kind of items and there need not be a causal relationship between them.

Modern functions vs relations

The modern idea of a function sounds a lot like a relation, but it needs to satisfy a special property:

A one-place – or unary – function from set \( A \) to set \( B \) is any binary relation \( R \) from \( A \) to \( B \) such that for all \( a \in A \), there is exactly one \( b \in B \) for which \( (a, b) \in R \).

We can break that down:

- There is always at least one \( b \in B \) with \( (a, b) \in R \).
- There is never more than one \( b \in B \) with \( (a, b) \in R \).
- The source \( A \) of the relation is its domain; every \( a \in A \) is the first term of some pair \( (a, b) \in R \).

Drawing a directed graph of a function, every point in the \( A \) circle has exactly one arrow going out from it to the \( B \) circle.
Examples: *Function or Not?*

Relations from $A$ to $B$, where

- $A = \{a, b, c, d\}$
- $B = \{1, 2, 3, 4, 5\}$

(i) $\{(a, 1), (b, 2), (c, 3)\}$?

**No.** $d \in A$ but there’s no pair for $d$ in the relation.

Examples: *Function or Not?*

Relations from $A$ to $B$, where

- $A = \{a, b, c, d\}$
- $B = \{1, 2, 3, 4, 5\}$

(ii) $\{(a, 1), (b, 2), (c, 3), (d, 4), (d, 5)\}$

**No.** $(d, 4) \in R$ and $(d, 5) \in R$, and $4 \neq 5$.

Examples: *Function or Not?*

Relations from $A$ to $B$, where

- $A = \{a, b, c, d\}$
- $B = \{1, 2, 3, 4, 5\}$

(iii) $\{(a, 1), (b, 2), (c, 3), (d, 5)\}$

**Yes!**

Examples: *Function or Not?*

Relations from $A$ to $B$, where

- $A = \{a, b, c, d\}$
- $B = \{1, 2, 3, 4, 5\}$

(iv) $\{(a, 1), (b, 2), (c, 2), (d, 1)\}$

**Yes!**
Examples: Function or Not?

Relations from $A$ to $B$, where

\[ A = \{a, b, c, d\} \]
\[ B = \{1, 2, 3, 4, 5\} \]

(v) \{(a, 5), (b, 5), (c, 5), (d, 5)\}

Yes! (This is a constant function)

$n$-place functions

We can generalize from these one-place functions to those that take $n$ inputs.

An $n$-place (or $n$-ary) function from sets $A_1, ..., A_n$ into $B$ is an $(n+1)$-place relation $R$ such that for all $a_1, ..., a_n$ with each $a_i \in A_i$, there is exactly one $b \in B$ with $(a_1, ..., a_n, b) \in R$.

When $n=1$, this is the same as a one-place (unary) function.

Alternative formulation of $n$-place functions

Alternatively, we can treat an $n$-place function from $A_1, ..., A_n$ into $B$ as a one-place function from the Cartesian product $A_1 \times ... \times A_n$ into $B$.

That would mean, e.g., having addition go from $f(x, y) = x + y$ to $f((x, y)) = x + y$, a one-place function from $N \times N$ into $N$.

Advantage of true $n$-place functions? Fewer parentheses.

Partial function

We can relax the definition of a function slightly to define a partial function from a set $A$ to a set $B$ as a binary relation $R$ from $A$ to $B$ such that for all $a \in A$, there is at most one $b \in B$ with $(a, b) \in R$.

That is, there may be elements in $A$ that have no pairs in the function.
Concept hierarchy

Relation

Partial function

Function

As we go down, we require more properties.

Operations on functions

Since functions are relations, all operations we applied to relations can be applied to functions.

E.g., we define *domain* and *range* the same way. Informally,

A function consists of pairs \((a, b)\)

The domain is the set of all \(a\)s.

The range is the set of all \(b\)s.

Functions: *Composition*

Can you think of an application where you want to call one function, \(g\), on the results of another function, \(f\)?

(That is, where the argument – or input – to \(f\) results in a value – or output – from \(f\), and that output is used as the input to \(g\), to get an output from \(g\)?)

This is extremely important in computer science!
If
\[ y = f(z) = z^2 \]
and
\[ z = g(x) = 2x - 1 \]
where \( y, z, \) and \( x \) are variables, such that \( y \) depends on \( z \) and \( z \) depends on \( x \), then \( y \) will depend on \( x \) and we may write:
\[ y = (2x - 1)^2 \]
or
\[ (f \circ g)(x) = (2x - 1)^2 \]

Definition of composition

Given functions
\[ f: A \to B \]
\[ g: B \to C \]
\[ g \circ f: A \to C = \{(a, c) \text{ such that there exists } x \text{ with } x = f(a) \text{ and } c = g(x)\} \]

When you compose two functions, the result is a function (since the domain of \( g \) is the same as the target of \( f \)).
Composition: Associativity
Composition of functions is associative.
Given
\[ f : A \to B \]
\[ g : B \to C \]
\[ h : C \to D \]
\[ (h \cdot (g \cdot f)) = ((h \cdot g) \cdot f) \]

Composition: Commutativity
Composition of functions is not commutative:
Given
\[ f : A \to B \]
\[ g : B \to C \]
\[ g \cdot f \text{ is a function but } f \cdot g \text{ is not a function unless } C = A. \]
Even if \( A = B = C \), \( g \cdot f \) and \( f \cdot g \) may be different functions.

Example: \( g \cdot f \) vs \( f \cdot g \)
\[ f(x) = x^2 + 3x - 2 \]
\[ g(x) = x - 1 \]
\( (f \cdot g)(x) = f[g(x)] = f(x - 1) = (x - 1)^2 + 3(x - 1) - 2 = x^2 + x - 4 \)
\( (g \cdot f)(x) = g[f(x)] = g(x^2 + 3x - 2) = (x^2 + 3x - 2) - 1 = x^2 + 3x - 3 \)
\( (g \cdot g)(x) = g(x - 1) = (x - 1) - 1 = x - 2 \)

Examples
For these functions \( f \) and \( g \), what is \( g \cdot f \)? What is \( f \cdot g \)?
\[ f(n) = n + 3 \]
\[ g(n) = 3n \]
\[ f(n) = (n + 1)^2 \]
\[ g(n) = n - 1 \]
\[ f = \{(1, 1), (2, 2), (3, 1)\} \]
\[ g = \{(1, 2), (2, 3), (3, 2)\} \]
Inverses

If \( y = f(x) \) is equivalent to \( x = g(y) \), the function \( g \) is said to be the inverse of \( f \) and \( f \) the inverse of \( g \).

We can denote these as \( f^{-1} \) and \( g^{-1} \):

\[
\begin{align*}
    f^{-1}[f(x)] &= x \\
    g^{-1}[g(y)] &= y
\end{align*}
\]

Example:

\[
\begin{align*}
    f(x) &= \sqrt{x}; \ x \geq 0 \\
    f^{-1}(x) &= x^2
\end{align*}
\]

We know the inverse of a relation is a relation; we just flip each pair.

Is the inverse of a function always a function?

Consider some examples where

- \( A = \{1, 2, 3\} \) and \( B = \{a, b, c, d\} \)
- \( f = \{(1, a), (2, d), (3, c)\} \). Is \( f^{-1} \) a function from \( B \) to \( A \)? Is it a function with some other domain and range?
- \( f = \{(1, a), (2, b), (3, b)\} \). Is \( f^{-1} \) a function from \( B \) to \( A \)? Is it a function with some other domain and range?