Assignments:

Assignment 3 due today.
Assignment 4 out later today.

Previously:

Introduce a special class of relations – functions!

Today:

More functions!

Injectivity

A function \( f : A \rightarrow B \) is injective (or one-to-one) iff whenever \( a \neq a' \), \( f(a) \neq f(a') \).

That is, iff distinct inputs lead to distinct outputs.
**Surjectivity**

A function $f: A \rightarrow B$ is surjective with respect to (or is onto) $B$ iff for all $b \in B$, there exists $a \in A$ with $f(a) = b$.

That is, every element of $B$ is the value of some element of $A$ under $f$; range($f$) = $B$.

**Bijectivity**

A function $f: A \rightarrow B$ that is both injective and onto $B$ is called a bijection between $A$ and $B$.

Equivalently, a function $f: A \rightarrow B$ is a bijection between $A$ and $B$ iff its inverse $f^{-1}$ is a function from $B$ to $A$.

**Can you spot the difference?**

Bijections are so orderly they make some functions look downright goofy.
Introduction to cardinality principles

Intuition: If a relation from $A$ to $B$ is many-to-one, that seems to say something about the relative sizes of its source $A$ and target $B$.

Relations are too general to support that intuition the way we want, but functions will work!

Notation: For any finite set $S$, $|S|$ is the number of elements in $S$. The textbook uses $\#(S)$.

The Principle of Equinumerosity

For finite sets $A$ and $B$, $|A| = |B|$ iff there is a bijective function $f: A \rightarrow B$.

$|A| = n = |B|$ so we can match each member of $A$ with the member of $B$ with the same number and vice versa.

– Recall $f: A \rightarrow B$ is injective iff for any $x, y \in A$, whenever $x \neq y$, $f(x) \neq f(y)$
– Recall $f: A \rightarrow B$ is surjective iff for all $b \in B$, there exists some $a \in A$ s.t. $f(a) = b$
– Recall $f: A \rightarrow B$ is bijective iff it is injective and surjective

The Principle of Comparison

For finite sets $A$ and $B$, $|A| \leq |B|$ iff there is an injective function $f: A \rightarrow B$

– Recall $f: A \rightarrow B$ is injective iff for any $x, y \in A$, whenever $x \neq y$, $f(x) \neq f(y)$

Diagram from Wikipedia

The Pigeonhole Principle

If there are more pigeons than holes for them to stay in, then at least one pigeonhole must have more than one pigeon in it.
The Pigeonhole Principle

For finite sets $A$ and $B$, if $|A| > |B|$, then no function $f: A \rightarrow B$ is injective.

Some value of $a \in A$ must share a $b$.

---

Pigeonhole example

A town has 400 inhabitants. Do any of them share a birthday?

$A =$ set of people in the village  
$B =$ set of days in the year  
$f: A \rightarrow B$ associates a villager with a birthday  
$|A| = 400 > 366 \geq |B|$  
Therefore, there is a $b \in B$ s.t. $f(a) = b$ for multiple $a$

---

Generalized Pigeonhole Principle

For finite sets $A$ and $B$,
if $|A| > k |B|$, then
for every function $f: A \rightarrow B$,
there is a $b \in B$ such that
$b = f(a)$
for at least $k + 1$ distinct $a \in A$.

---

Pigeonhole example

A town has 400 inhabitants. At least how many were born in a particular month?

$A =$ set of people in the village  
$B =$ set of months  
$f: A \rightarrow B$ associates a villager with a birth month  
$|A| = 400 > 396 = 33 |B|$  
Therefore, there is a month where at least 34 people were born.
Special functions

Identity functions

For any set $A$, the identity function on $A$ is the function $f: A \to A$ such that for all $a \in A$, $f(a) = a$.

Each set has a unique identity function.

To have a single identity function, it would need to be over a true universal set, and no such set can exist.

(Cf. the handout on Russell’s Paradox!)

Constant functions

For non-empty sets $A$ and $B$, a constant function on $A$ into $B$ is any function $f: A \to B$ such that $f(a) = f(a')$ for all $a, a' \in A$.

Equivalently, iff there is some $b \in B$ such that $b = f(a)$ for all $a \in A$.

Projection functions

For a function that takes two arguments $(f: A \times B \to C)$ we can fill in one of the arguments and what we get is a function that takes one argument:

right projection of $f$ from $a$ is $f_a: B \to C$

$f_a(b) = f(a, b)$ for every $b \in B$

left projection of $f$ from $b$: $f_b: A \to C$

$f_b(a) = f(a, b)$ for every $a \in A$
Curried functions

Projection functions are similar to the concept of Curried\(^1\) functions.

Currying translates the evaluation of a function that takes more than one argument into the evaluation of a sequence of unary functions.

This lets us use these functions in contexts — theoretical or practical — where functions can only take one argument.

\(^1\) Conceived by Moses Schönfinkel, 1924, and named for logician Haskell Curry (1900–1982).

Not Curried:
\[ f: A \times B \times C \rightarrow D \]
Curried:
\[ f: A \rightarrow (B \rightarrow (C \rightarrow D)) \]
\[ f(a): B \rightarrow (C \rightarrow D) \]
\[ f(a)(b): C \rightarrow D \]

Characteristic functions

A *characteristic function* for a set \( A \) specifies the truth-value of the statement \( u \in A \).

[A brief programming interlude]
Families and sequences

A family of sets refers to any function on a domain \( I \) (called an index set) such that for each \( i \in I \), \( f(i) \) is a set.

Writing \( f(i) \) as \( A_i \), it is thus the set of all ordered pairs \( (i, f(i)) = (i, A) \) with \( i \in I \).

The range of this function is the collection \( \{A_i | i \in I\} \).

Suppose the index set \( I \) has \( n \) elements:
\( I = \{1, \ldots, n\} \), but the function \( f \) is not injective, say \( f(1) = f(2) \), i.e., \( A_1 = A_2 \).

In this case, the family containing all the pairs \( (i, A) \) with \( i \leq n \), has \( n \) elements, but the collection, containing all the sets \( A_i \) with \( i \leq n \), has fewer elements since \( A_1 = A_2 \).

It’s convenient to identify an infinite sequence \( a_1, a_2, a_3, \ldots \) with a function \( f: \mathbb{N}^+ \to A \) for some appropriate set \( A \).

\( f(i) = a_i \) for each \( i \) in \( \mathbb{N}^+ \).

The \( i \)th term in the sequence is the value of the function for input \( i \).

So, the sequence is just a family with index set \( \mathbb{N}^+ \).