Previously:
Introduce probability

Today:
More probability:
  Independence
  Information and probability (Monty Hall problem)
  Verifying with simulations
Assignment 5 back
Assignment 6 due

Later:
More simulations
Conditional Independence

**Sample space**: A set $S$ of outcomes
Subsets of $S$ are called *events*

Example:
$$S = \{(0,0,0), (0,0,1), ..., (1,1,1)\}$$
A set of sequences of three coin tosses
Each triple is an outcome

Possible events are:
\begin{align*}
  \text{Got 2 heads: } & \{(0,1,1), (1,0,1), (1,1,0)\} \\
  \text{Got a head last: } & \{(0,0,1), (0,1,1), (1,0,1), (1,1,1)\}
\end{align*}

Notice that these two events happen to overlap.
In this example, each outcome is equally likely – but this isn’t always the case.
Addition principle:
Disjoint sets: \(|A \cup B| = |A| + |B|\)
Any sets: \(|A \cup B| = |A| + |B| - |A \cap B|\)

Example:
\(S = \{2, 3, 4, ..., 12\}\)
Potential results of tossing 2 dice
Possible events:
- Got an even number: \(\{2, 4, 8, 10, 12\}\)
- Got at least 7: \(\{7, 8, 9, 10, 11, 12\}\)
In this example, some outcomes are more likely than others, e.g., 7 is more likely than 2. Why?

Probability (density) function:
\(p: S \to [0,1]\)
Assigns a probability to each outcome in \(S\)
Sum of all \(p\) values must be 1
Can extend \(p\) to take arbitrary subsets of \(S\) (i.e., events) as its input:
\(p(E) = \text{sum of } p(s) \text{ values for all outcomes } s \text{ in the event } E\)
\(p(\emptyset) = 0\)
Example:
\(p(\text{two heads in three tosses}) = p(\{\text{HHT, HTH, THH}\}) = \frac{3}{8}\)

Sample space \(S\) together with a probability function \(p\) is called a probability space.
When every outcome has the same probability (i.e., when the probability function is uniform), then the probability of an event \(E\) is given by
\(p(E) = \frac{|E|}{|S|}\)

Additional properties of probability functions
\(p(S) = 1\)
\(p(S - A) = 1 - p(A)\)
Independence

Events $A$ and $B$ are called independent if and only if

$$p(A \cap B) = p(A) \cdot p(B)$$

Example:

Suppose you have a uniform distribution, then

$$p(A \cap B) = \frac{|A \cap B|}{|S|}$$
$$p(A) = \frac{|A|}{|S|}$$

If the ratio $|A| \div |S| = \frac{|A \cap B|}{|B|}$ then the probability of $A$ in these two contexts is the same; knowing that $B$ happened didn’t change the probability of $A$.

But then $|A| \cdot |B| = |A \cap B| \rightarrow p(A) \cdot p(B) = p(A \cap B)$.

The Monty Hall problem

Monty Hall hosted the gameshow *Let’s Make a Deal*, and one of the decisions contestants were asked to make is a classic thought experiment.
Three doors

Behind one door is a prize

Three doors

Behind two doors... not a prize.

Three doors

You select a door.

The host then opens one of the two doors you didn’t select to reveal a goat.

There’s always a goat, even if you picked one.

You are asked: “Would you like to switch to the other unopened door?”

Should you?
Although we don’t know which door has the prize, we’ve gained information, and the probability for the choices has changed.

Two scenarios:

You picked the right door.
- They can open either of the other doors to reveal a goat.

You picked one of the goats.
- There’s only one other door they can open to reveal a goat.

The host’s choice adds value to the door he didn’t open but doesn’t change what’s known about the one you initially picked.

Strategies

Don’t switch:
\[ p(W) = \frac{1}{3} \quad p(L) = \frac{2}{3} \]

Always switch:
- If you picked wrong and switch, you always win.
- If you picked right and switch, you always lose
- But you were more likely to pick a wrong door initially; switching improves your odds.
\[ p(W) = \frac{2}{3} \quad p(L) = \frac{1}{3} \]
[See 2018-11-14.scm]