Previously:
Used logic

Today:
Find out what logic is

Propositional Logic:
A Formal Language of Pure Thought

Historical notes

Logic goes back to... who?

Aristotle, that's who.
Aristotle (384–322 BCE) held that any logical argument could be reduced to two premises and a conclusion.

He gave three basic laws of logical reasoning, often referred to as classical or Aristotelian logic:

The principle of identity
A thing is itself; $A = A$.

The principle of the excluded middle
A proposition is either true or false; either $A$ or not $A$.

The principle of contradiction
No proposition can be both true and false; $A$ cannot be both $A$ and not $A$.

Principles of most modern logic are extensions of Aristotelian logic.

Gottfried von Leibniz (1646–1716) initiated the partnership of mathematics and logic.

He tried to find a lingua universalis – a language where errors in thinking would be equivalent to arithmetical errors.

This laid the foundation for symbolic logic.

George Boole (1815–1864) demonstrated that a system of algebra can be used to express logical relations.

Boolean algebra was originally devised as a system for logical reasoning but has had widespread applications.

Gottlob Frege (1848–1925) is the father of modern symbolic logic.


It is now known as predicate logic.
There’s more to the history of logic, but we’ll pick up later.

For now, let’s consider predicate logic.

Propositions

Propositions are the key elements to represent, analyze, or explain declarative knowledge.

A *proposition* (or *statement*) is a claim that has a *truth value*.

The *two-value principle* encompasses two kinds of truth values: *truth* and *falsity*.

No proposition is both true and false (Aristotle’s *principle of the excluded middle*)
Notation

Particular propositions are denoted by lowercase letters, such as $p$, $q$, $r$, and $s$.

Variables that stand for any proposition are Greek letters, such as $\alpha$ and $\beta$.

Example propositions

$p$: $2 + 5 = 7$
True!

$q$: New York City is the capital of New York
False!

Example non-propositions

$r$: $x + 5 = 8$

$r$ is not a proposition as the equation $x + 5 = 8$ is neither true nor false without substituting a number for $x$.

$s$: What time is it?

$s$ is not a proposition as this is a question, not a claim.

Logical connectives

Build new propositions out of old ones

Not, and, or, if, iff, ...

We’ll use symbols to show that these have well-defined meanings, like mathematical operations, rather than the ambiguous interpretations of the words.
Negation operator

The symbol denoting a negation is ¬.

¬p is read “not p” or “the negation of p”

Truth values:

If a proposition is false, then its negation is true.
If a proposition is true, then its negation is false.

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Conjunction operator

The conjunction symbol ∧ indicates “and”.

The expression p ∧ q is read “p and q”.

Truth values:

A conjunction is true only when both components are true:

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<tr>
<th>α</th>
<th>β</th>
<th>α ∧ β</th>
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Disjunction operator

The disjunction symbol ∨ indicates “or”

The expression p ∨ q is read “p or q”

Truth values:

A disjunction is true when at least one of its components is true:

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<tr>
<th>α</th>
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Inclusive vs exclusive disjunction

In words, “α or β” could mean:

Exclusive:

Host: “You may have coffee, or you may have tea.”
Guest: “Coffee, please.”

Inclusive:

Host: “Would you like cream or sugar?”
Guest: “Both, please.”
Inclusive vs exclusive disjunction

Context is important for disambiguation.

We tend to use exclusive or in everyday language.

 Logical or (\lor) is inclusive.

We tend to use inclusive or in mathematics.

Implication

The implication symbol \( \Rightarrow \) indicates “implies”

The expression \( p \Rightarrow q \) is read “\( p \) implies \( q \)” or “\( p \) then \( q \)”

\( p \Rightarrow q \)

\textit{hypothesis} \Rightarrow \textit{conclusion}

\textit{antecedent} \Rightarrow \textit{consequent}

Truth values:

An implication is false only when the antecedent is true and the conclusion is false:

<table>
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<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \alpha \Rightarrow \beta )</th>
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\( \Rightarrow \) vs “if...then...”

The relations between \textit{material implication} (the truth-functional \( \Rightarrow \)) and the everyday language “if...then...” are complex.

Everyday uses of “if...then...” usually assert much more than the truth-functional one does, e.g.,

Implicit generality
A temporal or causal link

Material implication in everyday language

A politician says, “If I am elected, then I will fix the environment”.

False is the speaker is elected doesn’t fix the environment
True if, e.g., the speaker doesn’t get elected.

“If today is Wednesday, then \( 2 + 2 = 4 \)”

True no matter what day it is.

“If today is Wednesday, then \( 2 + 2 = 5 \)”

True except on Wednesdays, even though \( 2 + 2 = 5 \) is false!
Converse and inverse

Implication:

Let $p$ be “It’s snowing”.
Let $q$ be “Class is cancelled”.
Then $p \Rightarrow q$ is “If it’s snowing, class is cancelled”.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
“If class is cancelled, then it’s snowing.”

The inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$.
“If it’s not snowing, then class is not cancelled.”

Logical equivalence

Two propositions are logically equivalent when they have true or false in the same rows of their truth tables.

The converse and inverse of a given implication have identical truth values and, consequently, are logically equivalent.

Converse and inverse

The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
The inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$.

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<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$q$</th>
<th>$\neg q$</th>
<th>$p \Rightarrow q$</th>
<th>$q \Rightarrow p$</th>
<th>$\neg p \Rightarrow \neg q$</th>
<th>$\neg q \Rightarrow \neg p$</th>
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The converse and inverse are phrased differently, but they have the same truth values.

Contrapositive

The contrapositive of $\alpha \Rightarrow \beta$ is $\neg \beta \Rightarrow \neg \alpha$.

Let $p$ be “It’s snowing”.
Let $q$ be “Class is cancelled”.
Then $p \Rightarrow q$ is “If it’s snowing, class is cancelled”.
And $\neg q \Rightarrow \neg p$ is “If class is not cancelled, it’s not snowing”.
**Contrapositive**

\[ p \implies q: \text{“If it’s snowing, class is cancelled.”} \]

\[ \neg q \implies \neg p: \text{“If class is not cancelled, it’s not snowing.”} \]

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<th>( p )</th>
<th>( \neg p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( p \implies q )</th>
<th>( q \implies p )</th>
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An implication and its contrapositive are logically equivalent.

**Comparing direct proof and proof by contraposition**

Consider numbers \( x \) and \( y \).

**Claim:** If both of \( x, y > 2 \), then \( x + y > 4 \)

**Direct proof:**
- Assume \( x > 2 \) and \( y > 2 \)
- Show \( x + y > 4 \)

**Contrapositive proof:**
- Assume it’s not the case that \( x + y > 4 \), i.e., assume \( x + y \leq 4 \)
- Show it’s not the case that both of \( x, y > 2 \), i.e., show at least one of \( x, y \leq 2 \)

Which of these proof strategies seems more intuitive?

**Comparing direct proof and proof by contraposition – the flip side**

Consider numbers \( x \) and \( y \).

**Claim:** If \( x + y < 4 \), then at least one of \( x, y \) is less than \( 2 \)

**Direct proof:**
- Assume \( x + y < 4 \)
- Show at least one of \( x, y < 2 \)

**Contrapositive proof:**
- Assume it’s not the case that at least one of \( x, y < 2 \), i.e., assume both \( x, y \geq 4 \)
- Show it’s not the case that \( x + y < 4 \), i.e., show \( x + y \geq 4 \)

Which of these proof strategies seems more intuitive?
Equivalence operator

The equivalence (or biconditional) symbol $\iff$ indicates “is equivalent to”.

- The expression $p \iff q$ is read “$p$ is equivalent to $q$” or “$p$ if and only if $q$”
- Also written as $=$

Equivalence operator

Truth values

$p \iff q$ is true exactly when both propositions have identical truth values.

In other words, $p \iff q$ is true if and only if $p \Rightarrow q$ and $q \Rightarrow p$.

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<tr>
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Properties of operators

Logical operators have an order of operations, just like mathematical operators.

- From low to high: negation, conjunction, disjunction, implication
- You can think of conjunction like multiplication and disjunction like addition
  - Math: $-k \cdot (x + y)$
  - Logic: $\neg p \land (q \lor r)$

Properties of operators

Disjunction and conjunction are **commutative** and **associative**.

- **Associative**: e.g., $p \land q \land r$ is $(p \land q) \land r$ is $p \land (q \land r)$
- **Commutative**: e.g., $p \land q$ is $q \land p$

Implication is right-associative.

$p \Rightarrow q \Rightarrow r$ is $p \Rightarrow (q \Rightarrow r)$
Exercise:
Evaluating boolean expressions

Let propositions

\( p \) be true
\( q \) be false
\( r \) be true

What do these expressions evaluate to?

\[ p \land \neg r \]
\[ p \implies q \]
\[ q \iff p \]
\[ r \lor (p \land q) \]
\[ (p \lor r) \implies ((p \lor q) \land r) \]

Tautologies

A tautology is a compound proposition that is true regardless of the truth values of its components.

From Greek tauta, “the same thing” and lógos, “a discourse” or “a reasoning”

Tautologies

The disjunction \( \alpha \lor \neg \alpha \), an application of the principle of the excluded middle, is a tautology.

It asserts that a proposition is either true or false – the two-valued principle; there are no values between true and false.

Example:

\( p \): “The book is in the library.”
\( p \lor \neg p \): “The book either is or is not in the library.” True!
**Principle of contradiction**

A proposition $\alpha \land \neg \alpha$ is always false.

Its negation, $\neg(\alpha \land \neg \alpha)$, is always true.

This is the principle of contradiction, stating that a proposition and its negation cannot be both true and false.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\neg \alpha$</th>
<th>$\alpha \land \neg \alpha$</th>
<th>$\neg(\alpha \land \neg \alpha)$</th>
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**Pitfalls**

*This statement is not true.*

Epimines’ paradox, named after the 6th c. BCE philosopher. If it’s true, it’s false. If it’s false, it’s true.

A variation:

The prisoner was told that by making a statement, he could choose the method of his execution; if the statement was true, he would be shot, and if false, he would be hanged. The prisoner made the statement, “I shall be hanged”.

**Definition**

**Basis:**

Every elementary letter is a formula.

**Recursion step:**

Whenever $\alpha$ is a formula, so is $\neg \alpha$.

Whenever $\alpha$, $\beta$ are formulas, so are $\alpha \land \beta$ and $\alpha \lor \beta$. 
Analyzing the definition

The definition tells us when a string is a formula. When is it not?

Implicit limiting clause: The only strings that are formulas are those that are required to be so by the basis and recursion step.

Implicit limiting clause in terms of sets

A string $\sigma$ is a formula iff it is an element of every set that satisfies both the basis and recursion clauses.

I.e., a string $\sigma$ is a formula iff it is an element of every set $S$ of strings that:
- contains all elementary letters
- is such that whenever $\alpha, \beta$ are in $S$, so are $(\neg \alpha)$, $(\alpha \land \beta)$, and $(\alpha \lor \beta)$

This is a top-down (non-constructive) formulation of the limiting clause; can also give a bottom-up formulation in terms of sequences.

A syntactic decomposition tree

```
\neg(p \lor \neg q) \land ((\neg r \land (p \lor (q \land s))) \land (\neg r \lor s))
```

Unique decomposition

Formulas are grammatically unambiguous:

Under our definition of a formula (with all the parentheses present), each formula has a unique syntactic decomposition tree.

Proof: By induction on the recursive definition of the set of formulas.
Inference

A way of going from certain statements... known as premises, assumptions, or hypotheses to another... known as the conclusion that preserves some desirable property.

Types of inference

Deductive inference

Preserve truth

Used in mathematics; our focus in this course

Non-deductive inference

Preserve probability / reliability / credibility

Used widely in everyday life, and increasingly in AI!
Intuitive notion of logical consequence

Consider inference with set of premises \( A = \{\alpha_1, \ldots, \alpha_n\} \) and conclusion \( \beta \).

\( A \) logically implies \( \beta \) iff it is impossible to have the disastrous combination where all \( \alpha_i \in A \) are true but \( \beta \) is false.

Write as
\[
\alpha_1, \ldots, \alpha_n \vdash \beta
\]

Logical consequence

Logical consequence is also called entailment or implication.

Written as \( A \vdash \beta \) (the textbook) or \( A \models \beta \) (elsewhere)

We’ll clarify this notion later.

But some structural properties already become apparent.

These are the Tarski conditions.

Tarski conditions

Conditions for being a closure relation:

Identity
\( A \vdash \alpha \) whenever \( \alpha \in A \)

Monotony
Whenever \( A \vdash \alpha \) and \( A \subseteq B \), then \( B \vdash \alpha \)

Cumulative transitivity
When \( A \vdash \beta \) for all \( \beta \in B \) and \( A \cup B \vdash \gamma \), then \( A \vdash \gamma \)

Rules of inference

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<tr>
<th>Name</th>
<th>LHS</th>
<th>RHS</th>
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<tbody>
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<td>Simplification</td>
<td>( \alpha \land \beta )</td>
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<tr>
<td>Conjunction</td>
<td>( \alpha, \beta )</td>
<td>( \alpha \land \beta )</td>
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<tr>
<td>Disjunction</td>
<td>( \alpha )</td>
<td>( \alpha \lor \beta )</td>
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The Forest of Sideways Logic

After the witch was killed, Hansel and Gretel wandered around in the forest trying to find their way home. At a fork in the path was stationed a forest ogre’s guard; one path lead out of the forest, and the other to the ogre’s supper table — and not as a guest!

By order of the ogre, the guard was to be asked only one question by each traveler or group of travelers. The dead witch’s grateful cat had warned Hansel and Gretel that the guard lied every other time. Hansel and Gretel walked, one by one, Hansel first, then Gretel.

What question did Gretel ask the guard to be sure to infer the correct direction?
The Forest of Sideways Logic

Gretel, going second, asked “What direction did you tell Hansel?”

The guard’s statement to Gretel is true only if the statement to Hansel was false, and false only if the statement to Hansel was true.

Consequently, if the reply to Gretel is “right”, then

“right” ⇒ the road to the left is the correct one
“left” ⇒ the road to the right is the correct one