Assignment 2
  Graded

Assignment 3
  Being graded!

Assignment 4
  Due now

Exam 1
  Due Monday Tuesday
Relationships

You’ve seen examples of relationships

between sets:

\[ \{1, 2\} \subseteq \{1, 2, 3\} \]

between numbers:

\[ 3 \leq 4 \]

between people:

Romeo loves Juliet

Since these relations focus on properties involving two objects, they are called *binary relations.*
Binary relations
10 < 12
$5 - (-2)$
5 \not< -2
aRb
$a \not\in R b$
Binary relations

A *binary relation over a set* $A$ is a predicate $R$ that can be applied to ordered pairs of elements drawn from $A$.

If $R$ is a binary relation over $A$ and it holds for the pair $(a, b)$, we write $aRb$.

$$3 = 3 \quad 5 < 7 \quad \emptyset \subseteq \mathbb{N}$$

If $R$ is a binary relation over $A$ and it does not hold for the pair $(a, b)$, we write $a\not{R}b$.

$$4 \neq 3 \quad 4 \not{<} 3 \quad \mathbb{N} \not{=} \emptyset$$
Properties of relations

Generally speaking, if $R$ is a binary relation over a set $A$, the order of the operands is significant.

For example, $3 < 5$, but $5 \nleq 3$.

In some relations, order is irrelevant – more on that later!

Relations are always defined relative to some underlying set.

It’s not meaningful to ask whether $🙂 \subseteq 15$, for example, because $\subseteq$ is defined over sets, not arbitrary objects.
For instance, \( \leq \) is a binary relation over \( \mathbb{N} \) because, given any \( a, b \in \mathbb{N} \), the following questions are meaningful and have definitive yes or no answers:

- Is \( a \leq b \) true?
- Is \( b \leq a \) true?
Visualizing relations

We can visualize a binary relation $R$ over a set $A$ by drawing the elements of $A$ and drawing an arrow between an element $a$ and an element $b$ if $aRb$ is true.
The relation $a | b$, meaning “$a$ divides $b$”, over the set $\{1, 2, 3, 4\}$ looks like this:
The relation $a \neq b$ over the set \{1, 2, 3, 4\} looks like this:
The relation $a = b$ over the set $\{1, 2, 3, 4\}$ looks like this:
Below is some relation over the set \{1, 2, 3, 4\}. There doesn’t appear to be a simple unifying rule, but that’s ok.

This is an isolated element. It doesn’t relate to anything, and nothing relates to it.
Below is some relation over the set \{1, 2, 3, 4\}. There doesn’t appear to be a simple unifying rule, but that’s ok.
Formal relations
Sets are good for asking questions about membership.

They’re not good when we care about the relative ordering of elements, e.g., for encoding

\[ 3 < 4 \]

but not

\[ 4 < 3 \]
When we care about the order of elements, we can use a *tuple*.

Tuples are the mathematical equivalent of lists in Racket. They can contain duplicates, and the order of elements is preserved.

A tuple of length 2 is called an *ordered pair*.
Sets:

\[ \{a, b\} = \{b, a\} \]

Tuples:

\[(a, b) \neq (b, a)\]
The *criterion for identity* of ordered pairs is

\[(x_1, x_2) = (y_1, y_2) \text{ iff } x_1 = y_1 \text{ and } x_2 = y_2.\]

For tuples of arbitrary length \(n\):

\[(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \text{ iff } x_i = y_i \text{ for all } i \text{ from 1 to } n.\]
Cartesian products

**DEFINITION.** If $A$ and $B$ are sets, the *Cartesian product* $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

The set of all ordered pairs whose first element is in $A$ and whose second element is in $B$.

$A \times A$ is also written $A^2$.

$|A \times B| = |A| \cdot |B|$
Cartesian?

After René Descartes. In the 17th century, he introduced the *Cartesian plane* we use in geometry, which is:

\[(x, y) \in \mathbb{R}^2.\]
Exercise

\[ A = \{a, b, c\} \]
\[ B = \{1, 2, 3\} \]
\[ C = \{\text{do, re, mi}\} \]

What’s \( A \times B \)?

What’s \( C \times A \)?
What if one of the sets is $\emptyset$?
What if one of the sets is $\emptyset$?

For any set $A$, $A \times \emptyset = \emptyset \times A = \emptyset$. 
Binary relations

DEFINITION. For any sets $A$ and $B$, a binary relation from $A$ to $B$ is a subset of the Cartesian product $A \times B$.

A relation is determined by the ordered pairs it covers, e.g.,

$L = \{(\text{Romeo, Juliet}),$
  \hspace{1em} \text{(Juliet, Romeo)},$
  \hspace{1em} \text{(Romeo, Rosaline)}^1,$
  \hspace{1em} \ldots\}$

^1 People forget about her. Romeo’s just in love with being in love. Tsk.
\[ n \text{-ary relations} \]

We can generalize from binary relations.

Let \( A_1, \ldots, A_n \) be sets (where \( n \geq 1 \)). An \( n \)-place relation over \( A_1 \times \cdots \times A_n \) is any subset of \( A_1 \times \cdots \times A_n \); it is a set of \( n \)-tuples.
Sources and targets, domains and ranges

When a relation is from $A$ to $B$, we call $A$ the *source* of the relation and $B$ the *target*.

But $A$ and $B$ aren’t unique for a relation – add elements to each to make $A'$ and $B'$ and then our relation is also a subset of $A' \times B'$.

The smallest source and target sets for a relation are called the *domain* and *range*.

- Domain of $R$: $\{a \mid (a, b) \in R \text{ for some } b\}$.
- Range of $R$: $\{b \mid (a, b) \in R \text{ for some } a\}$. 
Representing binary relations

\[ A = \{\text{John, Mary}\} \]

\[ B = \{\text{dog, cat, rabbit}\} \]

\[ R = \{(\text{John, dog}), (\text{John, rabbit}),
(\text{Mary, cat}), (\text{Mary, rabbit})\} \]

\( R \) could be the relation of \( a \) having an animal of type \( b \) as a pet.
Directed graphs

When we draw a graph of a relation, we can circle the sets for the domain and range.
Directed graphs

A directed graph $G$ is a pair $(V, E)$ where

- $V$ is a finite set of vertices (singular: vertex)
- $E$ is the set of edges – a binary relation on $V \times V$. 
Code
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