Binary Relations

Part 3

5 March 2020
Where are we?
Binary relations

A *binary relation over a set* $A$ is a predicate $R$ that can be applied to ordered pairs of elements drawn from $A$.

If $R$ is a binary relation over $A$ and it holds for the pair $(a, b)$, we write $aRb$.

\[ 3=3 \quad 5<7 \quad \emptyset \subseteq \mathbb{N} \]

If $R$ is a binary relation over $A$ and it does not hold for the pair $(a, b)$, we write $aRb$.

\[ 4\neq 3 \quad 4\not< 3 \quad \mathbb{N} \not\subseteq \emptyset \]
Binary relations

The formal definition of a binary relation is as a set of pairs.

That is, the \(<\) relation over the set \{1, 2, 3, 4\} is defined as the set of pairs where that predicate is true:

\[
\{(1, 2), (1, 3), (1, 4), \\
(2, 3), (2, 4), \\
(3, 4)\}
\]
Subrelations

Since binary relations are sets of ordered pairs, one relation can be a subset of another.

We call this being a \textit{subrelation}.
For example, *having a pet cat* is a subrelation of *having a pet*:

\[
\{(\text{👧}, \text{🐈}), (\text{🧓}, \text{🐈})\}\subseteq\{(\text{👧}, \text{🐈}), (\text{👩}, \text{🐇}), (\text{🧓}, \text{🐈}), (\text{🧔}, \text{🐟})\}\]
Additionally:

*brother-of* is a subrelation of *sibling-of*

* = is a subrelation of *≤*
Reflexivity

Some relations always hold from any element to itself. For example:

\[ x = x \text{ for any } x \]

\[ A \subseteq A \text{ for any set } A \]

These relations are called \textit{reflexive}.

\[ \forall a \in A \ . \ aRa \]

\textit{“Every element is related to itself”}
Reflexivity

∀a ∈ A . aRa

“Every element is related to itself”
Symmetry

For some relations, the relative order of the objects doesn’t matter. For example:

If \( x = y \) then \( y = x \).

If \( x \) is a sibling of \( y \), then \( y \) is a sibling of \( x \).

These relations are called symmetric.

\[ \forall a \in A . \forall b \in A . (aRb \Rightarrow bRa) \]

“If \( a \) is related to \( b \), then \( b \) is related to \( a \).”
Symmetry

∀a ∈ A . ∀b ∈ A . (aRb ⇒ bRa)

“If a is related to b, then b is related to a.”
Transitivity

Many relations can be chained together. For example:

If \( x = y \) and \( y = z \), then \( x = z \).

If \( R \subseteq S \) and \( S \subseteq T \), then \( R \subseteq T \).

These relations are called *transitive*.

\[
\forall a \in A \ . \ \forall b \in A \ . \ \forall c \in A \ . \ (aRb \land bRc \Rightarrow aRc)
\]

“Whenever \( a \) is related to \( b \) and \( b \) is related to \( c \), we know \( a \) is related to \( c \).”
Transitivity

∀a ∈ A . ∀b ∈ A . ∀c ∈ A . (aRb ∧ bRc ⇒ aRc)

“Whenever a is related to b and b is related to c, we know a is related to c.”
Equivalence relations

An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

For example:

- $x = y$
- $x$ lives in the same city as $y$
- $x$ has the same color as $y$
- $x$ has the same shape as $y$
\[ xRy \text{ if } x \text{ and } y \text{ have the same shape} \]
$xRy$ if $x$ and $y$ have the same shape
$x R y$ if $x$ and $y$ have the same shape
$xTy$ if $x$ and $y$ have the same color
$xTy$ if $x$ and $y$ have the same color
$x T y$ if $x$ and $y$ have the same color
$xSy$ if $x = y$
\[ xSy \text{ if } x = y \]
\[ xS_y \text{ if } x = y \]
Equivalence relations are a way of formalizing the idea of a partition of a set.
Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the equivalence class of $x$ is the set

$$[x]_R = \{y \in A \mid xRy\}$$

“The set of all elements related to $x$ by $R$”
$xRy \text{ if } x \text{ and } y \text{ have the same shape}$
$xRy$ if $x$ and $y$ have the same shape
xRy if x and y have the same shape
$x R y$ if $x$ and $y$ have the same shape
$x R y$ if $x$ and $y$ have the same shape
Prerequisite structures
CS courses

And so on
Basic Griddlecakes

Ingredients

2 cups sifted all purpose flour
1 tablespoon sugar
4 teaspoons baking powder
1 teaspoon salt
2 eggs, lightly beaten
1 1/2 cups milk
1/4 cup melted butter

Directions

Combine all the dry ingredients and sift into a mixing bowl. Combine the eggs, milk, and butter into the dry ingredients until the large lumps disappear. It is wise to remember that as this batter stands, it thickens. If this should happen while you are baking the cakes, add a little more milk, stirring it in with a wooden spoon or spatula. To bake, pour the batter on a hot, lightly greased griddle. The cakes should be about 4 to 6 inches in diameter. They are ready to turn when bubbles form and break and the edges seem cooked. Turn, and brown lightly on the reverse side. These are best served with melted butter and hot syrup or honey.

From James Beard’s American Cookery
Measure flour
Measure sugar
Measure baking powder
Measure salt
Combine dry ingredients
Heat griddle
Grease griddle
Melt butter
Measure milk
Beat eggs
Add wet ingredients
Cook pancakes
Relations and prerequisites

Imagine we have a prerequisite structure with no circular dependencies.

We can think about a binary relation $R$ where $aRb$ means “$a$ must happen before $b$”.

What properties of $R$ can we deduce from this?
Can this be the case?

Measure flour

must happen before

Measure flour
Can this be the case?

- Measure flour
- Combine dry ingredients
- Add wet ingredients

**must happen before**

**and**

**must happen before**

**therefore**

**must happen before**
Can this be the case?

Measure flour must happen before Combine dry ingredients

and

Combine dry ingredients must happen before Measure flour
\( aRa \)

\( aRb \land bRc \Rightarrow aRc \)

\( aRb \Rightarrow bRa \)
\[ \forall a \in A . \ aRa \]

\[ \forall a \in A . \ \forall b \in A . \ \forall c \in A . \ aRb \land bRc \Rightarrow aRc \]

\[ \forall a \in A . \ \forall b \in A . \ aRb \Rightarrow bRa \]
∀a ∈ A . aRa

∀a ∈ A . ∀b ∈ A . ∀c ∈ A . aRb ∧ bRc ⇒ aRc

∀a ∈ A . ∀b ∈ A . aRb ⇒ bRa
\[ \forall a \in A . \ aRa \]

Transitivity

\[ \forall a \in A . \forall b \in A . \ aRb \Rightarrow bRa \]
\[ \forall a \in A . \ aRa \]

Transitivity

\[ \forall a \in A . \ \forall b \in A . \ aRb \Rightarrow bRa \]
Irreflexivity

Some relations never hold from any element to itself.

For example, for any $x$, $x \not< x$.

These kind of relations are called *irreflexive*.

$$\forall a \in A . aRa$$

“No element is related to itself”
Irreflexivity

∀a ∈ A. aRa

“No element is related to itself”
Is this relation \textit{reflexive}?
Is this relation reflexive?

∀a ∈ A . aRa

“Every element is related to itself”
Is this relation reflexive?

∀a ∈ A. aRa

“Every element is related to itself”
Is this relation **irreflexive**?
Is this relation *irreflexive*?

∀a ∈ A . aRa

“No element is related to itself”
Is this relation *irreflexive*?

∀a ∈ A . aRa

“No element is related to itself”
Reflexivity and irreflexivity are *not* negations of one another!

Here’s the definition of reflexivity?

\[ \forall a \in A . \ aRa \]

What’s the negation of that statement?

\[ \exists a \in A . \ aRa \]

What’s the definition of irreflexivity?

\[ \forall a \in A . \ aRa \]
\[ \forall a \in A . \ aRa \]

Transitivity

\[ \forall a \in A . \forall b \in A . \ aRb \Rightarrow bRa \]
Irreflexivity

∀ a ∈ A . ∀ b ∈ A . aRa \implies bRa

Transitivity

∀ a ∈ A . ∀ b ∈ A . aRb \implies bRa
Irreflexivity

\[ \forall a \in A . \forall b \in A . a R b \implies b R a \]
Asymmetry

In some relations, the relative order of the objects can never be reversed.

For example, if \( x < y \), then \( x \not< x \).

These kind of relations are called **asymmetric**.

\[
\forall a \in A \ . \ \forall b \in A \ . (aRb \Rightarrow aRa)
\]

“If a relates to b, then does not relate to a.”
Asymmetry

∀a ∈ A . ∀b ∈ A . (aRb ⇒ aRa)

“If a relates to b, then does not relate to a.”
Are symmetry and asymmetry negations of one another?
Irreflexivity

Transitivity

\[ \forall a \in A . \forall b \in A . aRb \Rightarrow bRa \]
Irreflexivity

Transitivity

Asymmetry
A **strict order** is a relation that is irreflexive, asymmetric, and transitive.

For example:

\[ x < y \]

\[ a \text{ can run faster than } b \]

\[ A \subset B \]

**Strict orders are useful for**

- representing prerequisite structures,
- modeling dependencies,
- listing preferences.
Strict order proofs
Let’s suppose you’re asked to prove that a binary relation is a strict order.

Using the definition, you could prove that the relation is:

- asymmetric, 
- irreflexive, and
- transitive.

However, there’s a somewhat easier approach we can use instead!
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.
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What’s the high-level structure of this proof?
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$\forall R . (\text{Asymmetric}(R) \Rightarrow \text{Irreflexive}(R))$
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What’s the high-level structure of this proof?

$\forall R . (\text{Asymmetric}(R) \Rightarrow \text{Irreflexive}(R))$

Therefore we’ll choose an arbitrary asymmetric relation $R$ and prove that $R$ is irreflexive.
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

What’s the high-level structure of this proof?

$\forall R . (\text{Asymmetric}(R) \Rightarrow \text{Irreflexive}(R))$

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PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.

What's the definition of irreflexivity?
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.

What’s the definition of irreflexivity?

$$\forall x \in A . \ xRx$$
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.

What’s the definition of irreflexivity?

\[ \forall x \in A \cdot xRx \]

What’s the negation of this?
THEOREM. Let \( R \) be a binary relation over a set \( A \). If \( R \) is asymmetric, then \( R \) is irreflexive.

PROOF. Let \( R \) be an arbitrary asymmetric binary relation over a set \( A \). We will prove that \( R \) is irreflexive.

To do so, we will proceed by contradiction.

What’s the definition of irreflexivity?

\[ \forall x \in A . x R x \]

What’s the negation of this?

\[ \exists x \in A . x R x \]
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction.

What’s the definition of irreflexivity?

$$\forall x \in A . xRx$$

What’s the negation of this?

$$\exists x \in A . xRx$$

So let’s suppose that there is some element $x \in A$ such that $xRx$ and proceed from there.
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$. 
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds.
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds.
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds. However, this is impossible, since we can’t have both $xRx$ and $xRx$. 
THEOREM. Let $R$ be a binary relation over a set $A$. If $R$ is asymmetric, then $R$ is irreflexive.

PROOF. Let $R$ be an arbitrary asymmetric binary relation over a set $A$. We will prove that $R$ is irreflexive.

To do so, we will proceed by contradiction. Suppose that $R$ is not irreflexive. That means that there must be some $x \in A$ such that $xRx$.

Since $R$ is asymmetric, we know for any $a, b \in A$ that if $aRb$ holds, then $bRa$ holds. Plugging in $a=x$ and $b=x$, we see that if $xRx$ holds, then $xRx$ holds. We know by assumption that $xRx$ is true, so we conclude that $xRx$ holds. However, this is impossible, since we can’t have both $xRx$ and $xRx$.

We have reached a contradiction, so our assumption must have been wrong. Thus $R$ must be irreflexive. ■
THEOREM. If a binary relation $R$ is asymmetric and transitive, then $R$ is a strict order.

PROOF. Let $R$ be a binary relation that is asymmetric and transitive. Since $R$ is asymmetric, by our previous theorem we know that $R$ is also irreflexive. Therefore, $R$ is asymmetric, irreflexive, and transitive, so by definition $R$ is a strict order. ■

To prove that a binary relation is a strict order, just prove that it’s asymmetric and transitive.
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