Functions

Part 1

23 March 2020
‘I wish it need not have happened in my time,’ said Frodo.

‘So do I,’ said Gandalf, ‘and so do all who live to see such times. But that is not for them to decide. All we have to decide is what to do with the time that is given us.’

Where are we?
Logic for giving formal, precise definitions.

Mathematical proof for giving a convincing argument that a claim is true.

Sets for representing collections of entities.

Tuples for when order matters.

Relations express properties involving two or more entities, formalized as sets of tuples.
Let $E$ be the “eats” relation:

- elephants $E$ grass  \((\text{elephants}, \text{grass}) \in E\)
- elephants $E$ fruit  \((\text{elephants}, \text{fruit}) \in E\)
- elephants $E$ tree bark  \((\text{elephants}, \text{tree bark}) \in E\)

...
Let $E$ be the “eats” relation:

- anteaters $E$ insects
  
  (anteaters, insects) $\in E$

- spiders $E$ insects
  
  (spiders, insects) $\in E$

- pangolins $E$ insects
  
  (pangolins, insects) $\in E$

  …
However:

Each object in a store only has one price.

Each Vassar student has one 999 number.

These are examples of an important kind of relation, which gets its own name: a function.
What is a function?
Functions,
high-school edition
\[ f(x) = x^4 - 5x^2 + 4 \]
\[ f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21} \]
Functions, high-school edition

In high school, functions are usually given as equations like $f(x) = x^2$.

What does a function do?

- It takes as input a real number.
- It outputs a real number except when there are vertical asymptotes or other discontinuities, in which case the function doesn’t output anything.
Example

If we buy coffee and pay by the mass, we can say that the price is a function of the mass.

If coffee is $9/kg, half a kg is $4.50, 2 kg is $18, etc.

That is, we have the formula

\[ f(x) = 9x \]

where \( x \) designates the purchased amount (kg) and \( f(x) \) is the price (in $). Here, \( f(x) \) is defined only for \( x \geq 0 \).
Functions,
computer science edition
(define (flip-until num-heads (tries 0))
  (cond [(= num-heads 0) tries]
        [ (= (random 2) 1) (flip-until (sub1 num-heads) (add1 tries))]
        [else (flip-until num-heads (add1 tries))]))

> (flip-until 10)
29
(define (flip-until num-heads (tries 0))
  (cond [ (= num-heads 0) tries]
        [ (= (random 2) 1) (flip-until (sub1 num-heads) (add1 tries))]
        [else (flip-until num-heads (add1 tries))]))

> (flip-until 10)
29

How many times we needed to flip a coin to get 10 heads
(define (flip-until num-heads (tries 0))
  (cond
   [(= num-heads 0) tries]
   [(= (random 2) 1)
    (flip-until (sub1 num-heads)
               (add1 tries))]
   [else
    (flip-until num-heads
               (add1 tries))])))

> (flip-until 10)
29

*This is a cool shorthand to give a default value for a parameter*

*How many times we needed to flip a coin to get 10 heads*
Functions, computer science edition

In programming, functions

might take in inputs,
might return values,
might have side effects,
might never return anything,
might crash, and
might return different values when called multiple times.
What do these have in common?

They take in inputs.

They produce outputs.
A function is an object $f$ that takes in exactly one input, $x$, and produces exactly one output, $f(x)$. 

Rough idea of a function
Rough idea of a function

A function is an object $f$ that takes in exactly one input, $x$, and produces exactly one output, $f(x)$.

This isn’t a complete definition; we’ll come back to that.
High-school vs CS functions

In high school, functions were usually given by a rule like

\[ f(x) = 4x + 15 \]

In CS, functions are usually given by code like

\[ (\text{define (factorial n)}) \]
\[ (\text{if (= n 1)} \]
\[ 1 \]
\[ 1 \]
\[ (* n (\text{factorial (- n 1)}))))) \]

What sort of functions will we allow from a theoretical perspective?
...but also...
\[ f(x) = x^2 + 3x - 15 \]
$$f(n) = \begin{cases} 
-n/2 & \text{if } n \text{ is even} \\
(n+1)/2 & \text{otherwise}
\end{cases}$$

Functions like these are called \textit{piecewise functions}.
To define a function, you will typically either draw a picture, or give a rule for determining the output.
In theoretical computer science, a function:

Can have any kind of objects as the input and output, not just numbers.

Doesn’t need to express a causal relationship between the input and output.
Additionally, our functions are *deterministic*.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of Racket code, but it’s not a valid function under our definition:

```racket
(define (maybe-invert x)
  (if (= (random 2) 0)
      (- x)
      (- x) x))
```
One challenge
\[ f(x) = x^2 + 2x + 5 \]
\[ f(x) = x^2 + 2x + 5 \]
\[ f(3) = 3^2 + 2 \cdot 3 + 5 = 20 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 2 \cdot 3 + 5 = 20 \]

\[ f(0) = 0^2 + 2 \cdot 0 + 5 = 5 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 2 \cdot 3 + 5 = 20 \]

\[ f(0) = 0^2 + 2 \cdot 0 + 5 = 5 \]

\[ f(\text{🐰}) = \ldots? \]
\[ f(\text{ Pikachu }) = \text{ Pikachu} \]

\[ f(137) = \ldots ? \]
We need to make sure we can’t apply functions to meaningless inputs.
Domains and codomains

Every function $f$ has two sets associated with it: its *domain* and its *codomain*.

A function $f$ can only be applied to elements of its domain.
For any $x$ in the domain, $f(x)$ belongs to the codomain.
Domains and codomains

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\textbf{The function must be defined for every element of the domain.}
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A function $f$ can only be applied to elements of its domain. 

For any $x$ in the domain, $f(x)$ belongs to the codomain.

The function must be defined for every element of the domain.

The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.
Domains and codomains

Every function $f$ has two sets associated with it: its \textit{domain} and its \textit{codomain}.

A function $f$ can only be applied to elements of its domain.

For any $x$ in the domain, $f(x)$ belongs to the codomain.

\textbf{The domain of this function is $\mathbb{R}$. Any real number can be provided as input.}

\textbf{The codomain of this function is $\mathbb{R}$.}

\textbf{Every real number produced is a real number, though not all real numbers can be produced.}

```scheme
;; absolute-value : Number -> Number
;; Return the distance of $x$ from 0
(define (absolute-value x)
  (if (>= x 0)
      x
      (- x)))
```
Domains and codomains

If $f$ is a function whose domain is $A$ and whose codomain is $B$, we write $f : A \rightarrow B$.

In Racket, we write these as comments.

In Java (and C/C++), it’s explicit in the code:

```
B f(A arg) {
...
}
```

is equivalent to

```
f : A \rightarrow B
```
The rules for functions

We say that \( f : A \to B \) if these rules hold:

1. \( f \) must obey its domain/codomain rules:

\[
\forall a \in A . \, \exists b \in B . \, f(a) = b
\]

"Every input in \( A \) maps to some output in \( B \)."

2. \( f \) must be deterministic:

\[
\forall a_1 \in A . \, \forall a_2 \in A . \, (a_1 = a_2 \Rightarrow f(a_1) = f(a_2))
\]

"Equal inputs produce equal outputs."
Defining functions

Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.

Examples:

\[ f(n) = n + 1, \text{ where } f : \mathbb{Z} \rightarrow \mathbb{Z} \]
\[ g(x) = \sin x, \text{ where } g : \mathbb{R} \rightarrow \mathbb{R} \]

Notice that we’re giving both a rule and the domain/codomain.
Defining functions

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Notice that we’re giving both a rule and the domain/codomain.
Is this a function from $A$ to $B$?
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$A$: California, Delaware, New York, Washington, DC

$B$: Albany, Dover, Sacramento
Is this a function from $A$ to $B$?
Functions are a special kind of relation

Suppose A and B are sets. A function $f$ from A to B ($f : A \rightarrow B$) is a binary relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$, the relation $f$ contains exactly one ordered pair of the form $(a, b)$. The statement $(a, b) \in f$ is abbreviated $f(a) = b$. 
Functions: Composition
People

Ann
Andy
April
Leslie
Ron
People

Ann
Andy
April
Leslie
Ron

Places

Albany, NY
New York, NY
Poughkeepsie, NY
Rochester, NY
People

Ann
Andy
April
Leslie
Ron

Places

Albany, NY
New York, NY
Poughkeepsie, NY
Rochester, NY

\( f : \text{People} \rightarrow \text{Places} \)
April
Andy
Leslie
Ron

Ann
Albany, NY
New York, NY
Poughkeepsie, NY
Rochester, NY

A king’s ransom
A modest amount
More than you’d expect

People
Places
Prices

\( f : \text{People} \rightarrow \text{Places} \)
Ann
Andy
April
Leslie
Ron

Albany, NY
New York, NY
Poughkeepsie, NY
Rochester, NY

A king’s ransom
A modest amount
More than you’d expect

People → Places → Prices
\[ h : \text{People} \rightarrow \text{Prices} \]
\[ h(x) = g(f(x)) \]

\[ f : \text{People} \rightarrow \text{Places} \]
\[ g : \text{Places} \rightarrow \text{Prices} \]
**People**
- Ann
- Andy
- April
- Leslie
- Ron

**Places**
- New York, NY
- Poughkeepsie, NY
- Rochester, NY
- Albany, NY

**Prices**
- A king's ransom
- A modest amount
- More than you'd expect

**Functions**
- \( f : \text{People} \rightarrow \text{Places} \)
- \( g : \text{Places} \rightarrow \text{Prices} \)
- \( h : \text{People} \rightarrow \text{Prices} \)

**Equation**
- \( h(x) = g(f(x)) \)
$h : \text{People} \rightarrow \text{Prices}$

$h(x) = g(f(x))$

$f : \text{People} \rightarrow \text{Places}$

$g : \text{Places} \rightarrow \text{Prices}$

- Ann
- Andy
- April
- Leslie
- Ron
- Albany, NY
- New York, NY
- Poughkeepsie, NY
- Rochester, NY

A king's ransom
A modest amount
More than you'd expect
$h : \text{People} \rightarrow \text{Prices}$

$h(x) = g(f(x))$
Function composition

Suppose we have two functions, \( f : A \rightarrow B \) and \( g : B \rightarrow C \).

Notice that the codomain of \( f \) is the domain of \( g \). This means that we can use the outputs from \( f \) as inputs to \( g \).
Function composition

Suppose we have two functions, \( f : A \to B \) and \( g : B \to C \).

The \textit{composition of \( f \) and \( g \)}, denoted \( g \circ f \), is a function where

\[
g \circ f : A \to C, \quad \text{and} \quad (g \circ f)(x) = g(f(x)).
\]

The name of the function is \( g \circ f \), but when we apply it to an input \( x \), we write \( (g \circ f)(x) \). I don’t know why, but that’s what we do!
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 1 \) and \( g : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( g(n) = n^2 \).

What is \( g \circ f \)?
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 1 \)
and \( g : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( g(n) = n^2 \).

What is \( g \circ f \)?

\[
(g \circ f)(n) = g(f(n))
\]
Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = n^2$.

What is $g \circ f$?

$$(g \circ f)(n) = g(f(n))$$

$$= g(2n + 1)$$
Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = n^2$.

What is $g \circ f$?

$$(g \circ f)(n) = g(f(n))$$

$= g(2n + 1)$$

$= (2n + 1)^2 = 4n^2 + 4n + 1$$
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 1 \) and \( g : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( g(n) = n^2 \).

What is \( g \circ f \)?

\[
(g \circ f)(n) = g(f(n)) \\
= g(2n + 1) \\
= (2n + 1)^2 = 4n^2 + 4n + 1
\]

What is \( f \circ g \)?
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 1 \)
and \( g : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( g(n) = n^2 \).

What is \( g \circ f \)?

\[
(g \circ f)(n) = g(f(n)) \\
= g(2n + 1) \\
= (2n + 1)^2 = 4n^2 + 4n + 1
\]

What is \( f \circ g \)?

\[
(f \circ g)(n) = f(g(n))
\]
Let $f: \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g: \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = n^2$.

What is $g \circ f$?

$$(g \circ f)(n) = g(f(n))$$

$$= g(2n + 1)$$

$$= (2n + 1)^2 = 4n^2 + 4n + 1$$

What is $f \circ g$?

$$(f \circ g)(n) = f(g(n))$$

$$= f(n^2)$$
Let $f: \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g: \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = n^2$.

What is $g \circ f$?

\[
(g \circ f)(n) = g(f(n))
\]

\[
= g(2n + 1)
\]

\[
= (2n + 1)^2 = 4n^2 + 4n + 1
\]

What is $f \circ g$?

\[
(f \circ g)(n) = f(g(n))
\]

\[
= f(n^2)
\]

\[
= 2n^2 + 1
\]
Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function $f$ is evaluated first.

This is analogous to composing function calls in Racket:

If you write $(\text{sqr} \ (\text{first} \ l))$, you evaluate $(\text{first} \ l)$, and then you evaluate $\text{sqr}$ with the output of $(\text{first} \ l)$ as its input.
So, function composition is not *commutative*:

\[ f \circ g \text{ is not, in general, the same as } g \circ f. \]

However, function composition is *associative*:

\[ h \circ (g \circ f) = (h \circ g) \circ f \]
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