These notes include a skeleton framework for an example *structural induction* proof, a proof that all propositional logic expressions (*PLEs*) contain an even number of parentheses. (Please note that this is simply a skeleton framework, with details remaining to be filled in for a full proof!)

Recall that structural induction is a method for proving statements about recursively defined sets. To show that a property $P$ holds for all elements of a recursively defined set:

**Base Case(s)** Show that $P$ holds for every element in the basis for the recursive definition.

**Inductive Case(s)** Show that every constructor in the definition preserves property $P$.

For example, consider our recursive definition of propositional logic expressions:

**Base** Given an initial set $A$ of propositional letters (e.g., $p, q, r, \ldots$), all elements of $A$ are *PLEs*.

**Induction** If $P, Q$ are *PLEs*, then the following are also *PLEs* (note that the parentheses are part of the expressions):

- $(\neg P)$
- $(P \land Q)$
- $(P \lor Q)$
- $(P \rightarrow Q)$
- $(P \leftrightarrow Q)$

Note that the capital letters $P, Q$ are not propositional letters (which are lowercase)—they are variables that stand for propositional logic expressions. (For example, $P$ could stand for $((p \lor q) \land r)$, so $(\neg P)$ would be $((p \lor q) \land r)$, etc. In general, $P, Q$ can stand for any propositional logic expressions.)

Below is the framework of a proof that a property $P$ applies to all *PLEs*, where property $P$ is “*PLE* $P$ contains an even number of parentheses.”

Consider the claim:

**To prove:** All propositional logic expressions contain an even number of parentheses. (We consider 0 to be an even number.)

We prove the Claim holds for all *PLEs* by structural induction—first, we show it holds for all propositional letters (the base case), and then we show that each of the five constructors of *PLEs* preserves the truth of it; thus, because all *PLEs* are either propositional letters or the result of applying a constructor to existing *PLEs*, we have shown it holds for all *PLEs*. (That’s structural induction!)
Proof. We start with the Base Case: For every propositional letter \( p \), \( p \) has 0 parentheses, which we consider to be an even number.

For the Inductive Case, we have five sub-proofs. Because all five constructors can be seen as building new PLEs from at most two existing PLEs, we will use \( P \) and \( Q \) to stand for arbitrarily chosen PLEs in the cases that follow. (What should the inductive hypothesis be?) The cases correspond to the constructors:

1. **Constructor** \((\neg P)\) For this, the I.H. is that \( P \) has an even number of parentheses, and we show that under that hypothesis, \((\neg P)\) does, as well. Let \( n \) stand for the number of parentheses in \( P \). Then . . . (how would you show that \((\neg P)\) also has an even number of parentheses?)

2. **Constructor** \((P \land Q)\) For this, the I.H. is that each of \( P \) and \( Q \) has an even number of parentheses, and we show that under that hypothesis, \((P \land Q)\) does, as well. For notation, let \( n_P \) stand for the number of parentheses in \( P \) and \( n_Q \) stand for the number of parentheses in \( Q \). Then . . . (how would you show that \((P \land Q)\) also has an even number of parentheses?)

3. **Constructor** \((P \lor Q)\) . . . (this is very similar to case 2 above, although a full proof would include it in detail).

4. **Constructor** \((P \to Q)\) . . . (this is very similar to case 2 above, although a full proof would include it in detail).

5. **Constructor** \((P \leftrightarrow Q)\) . . . (this is very similar to case 2 above, although a full proof would include it in detail).

Having now proved that property \( P \) holds for every base case (the propositional letters) and that each constructor preserves \( P \)—that is, if \( P \) is true for the things on which the constructors operate, then \( P \) is true for the result of the constructor—this above proof by structural induction shows that \( P \) holds for every PLE.