Submissions due: 23 November, 1:30 p.m.
Corrections due: 30 November, 1:30 p.m.

Exercise 1

We've introduced several terms and definitions related to Turing machines, languages, and what it means to solve a problem. Some of these terms can only describe Turing machines, while others can only describe languages. A statement that uses terms incorrectly has no meaning; it cannot be evaluated as being true or false.

Consider, by analogy, the statement “the set $\mathbb{N}$ is even”. This statement doesn't type-check because “even” only applies to individual numbers, and $\mathbb{N}$ isn't a number. Similarly, the statement “$1 \in 5$” doesn't type-check because 5 isn't a set.

On the other hand, the statement $\mathbb{Z} \subseteq \mathbb{N}$ type-checks, but it isn't true – the integers aren't a subset of the natural numbers.\footnote{This is the mathematical equivalent of a grammatically correct sentence that's false, like “Otters aren't cute”. You understand what the sentence means, but it's objectively not true. They're adorable.}

For each of the following statements, decide whether
\begin{itemize}
  \item the statement type-checks and is true,
  \item the statement type-checks and is false, or
  \item the statement does not type-check.
\end{itemize}
Briefly explain your answers.

a. If $M$ is a Turing machine, $w$ is a string, and $M$ accepts $w$, then $A_{TM}$ accepts $\langle M, w \rangle$.

b. If $M$ is a Turing machine, $w$ is a string, and $M$ loops on $w$, then $\langle M, w \rangle \notin L(U)$. 
c. $U$ is decidable.

d. $\langle U \rangle$ is decidable.

e. $\{\langle U \rangle\}$ is decidable.
For Exercises 2 and 3, you should write proofs by construction, where you define Turing machines using the “high-level description” format introduced in class on Thursday and seen in the textbook, e.g., in the solutions to problems 4.10–4.14 on pages 213–4.

Exercise 2

Prove that the class of decidable languages (R) is closed under the intersection operation.
Exercise 3

Prove the following language is decidable:

\[ L = \{ (M, n) \mid M \text{ is a Turing machine with } n \text{ states, and there exists a string } w \in \Sigma^* \text{ of length at most } n^2 \text{ such that } M \text{ accepts } w \text{ in at most } n^3 \text{ transitions} \} \]

The choices of \( n^3 \) as the length limit and \( n^3 \) as the number of transitions are arbitrary. Your proof should require almost no modification if I'd chosen \( n \) or \( 2n \) or \( 100^n \).