Exercise 1

We’ve introduced several terms and definitions related to Turing machines, languages, and what it means to solve a problem. Some of these terms can only describe Turing machines, while others can only describe languages. A statement that uses terms incorrectly has no meaning; it cannot be evaluated as being true or false.

Consider, by analogy, the statement “the set \( \mathbb{N} \) is even”. This statement doesn’t type-check because “even” only applies to individual numbers, and \( \mathbb{N} \) isn’t a number. Similarly, the statement “\( 1 \in 5 \)” doesn’t type-check because 5 isn’t a set.

On the other hand, the statement \( \mathbb{Z} \subseteq \mathbb{N} \) type-checks, but it isn’t true – the integers aren’t a subset of the natural numbers.\(^1\)

For each of the following statements, decide whether
- the statement type-checks and is true,
- the statement type-checks and is false, or
- the statement does not type-check.

Briefly explain your answers.

a. If \( M \) is a Turing machine, \( w \) is a string, and \( M \) accepts \( w \), then \( A_{TM} \) accepts \( \langle M, w \rangle \).

\(^1\) This is the mathematical equivalent of a grammatically correct sentence that’s false, like “Otters aren’t cute”. You understand what the sentence means, but it’s objectively not true. They’re adorable.
b. If $M$ is a Turing machine, $w$ is a string, and $M$ loops on $w$, then $\langle M, w \rangle \notin L(U)$.

c. $U$ is decidable.

d. $\langle U \rangle$ is decidable.

e. $\{ \langle U \rangle \}$ is decidable.
For Exercises 2 and 3, follow the style of proof by construction seen in the textbook in sections 4.1 and 4.2. In particular, you may want to review the solutions to problems 4.10, 4.12, and 4.14 on pages 213–4.

Exercise 2

Prove that the collection of decidable languages (R) is closed under the intersection operation.
Exercise 3

Prove the following language is decidable:

\[ L = \{ (M, n) \mid M \text{ is a Turing machine with } n \text{ states, and there exists a string } w \in \Sigma^* \text{ of length at most } n^2 \text{ such that } M \text{ accepts } w \text{ in at most } n^3 \text{ transitions} \} \]

The choices of \( n^3 \) as the length limit and \( n^3 \) as the number of transitions are arbitrary. Your proof should require almost no modification if I'd chosen \( n \) or \( 2n \) or \( 100 \).