Exercise 1

We've introduced several terms and definitions related to Turing machines, languages, and what it means to solve a problem. Some of these terms can only describe Turing machines, while others can only describe languages. A statement that uses terms incorrectly has no meaning; it cannot be evaluated as being true or false.

Consider, by analogy, the statement “the set \( \mathbb{N} \) is even”. This statement doesn’t type-check because “even” only applies to individual numbers, and \( \mathbb{N} \) isn’t a number. Similarly, the statement “1 ∈ 5” doesn’t type-check because 5 isn’t a set.

On the other hand, the statement \( \mathbb{Z} \subseteq \mathbb{N} \) type-checks, but it isn’t true – the integers aren’t a subset of the natural numbers.\(^1\)

For each of the following statements, decide whether

- the statement type-checks and is true,
- the statement type-checks and is false, or
- the statement does not type-check.

Briefly explain your answers.

a. If \( M \) is a Turing machine, \( w \) is a string, and \( M \) accepts \( w \), then \( A_{\text{TM}} \) accepts \( \langle M, w \rangle \).
b. If $M$ is a Turing machine, $w$ is a string, and $M$ loops on $w$, then $(M, w) \notin L(U)$.

c. $U$ is decidable.

d. $(U)$ is decidable.

e. $\{(U)\}$ is decidable.
Exercise 2

Below are examples of Python functions that act as recognizers or deciders. For each of the functions,

- What is the language the function recognizes? Use set-builder notation if appropriate.
- Is the function a decider or just a recognizer? Why?

For this exercise, let \( \Sigma = \{a, b\} \), meaning that you only need to consider input strings made of as and bs.

a. Answer the questions above about this function:

```python
def arthur(w: str) -> bool:
    length = len(w)

    if length % 2 == 1:
        return False

    # Indices within strings range from 0 (the first character) to length - 1 (the last character).
    for i in range(0, length / 2):
        if (w[i] != "a") or (w[length - 1 - i] != "b"):
            return False

    return True
```

\[ \text{\% is the modulus operator, which gives the remainder after dividing the first number by the second.} \]

\[ \text{\footnote{If you haven't used Python before, don't panic! You can basically treat them as simplified Java, but feel free to ask questions about any parts you don't understand.}} \]
b. Answer the questions above about this function:

```python
def zaphod(w: str) -> bool:
    strings = ["""]

    while True:
        next_strings = []

        for x in strings:
            if w == x:
                return True
            next_strings.append(x + "a")
            next_strings.append(x + "b")

        # On the next iteration through the `while` loop,
        # use next_strings for the `for` loop.
        strings = next_strings
```

c. Answer the questions above about this function:

```python
def trillian(w: str) -> bool:
    x = 0
    y = 1
    while len(w) != x:
        z = x + y
        x = y
        y = z
    return True
```
Exercise 3

This exercise explores the following fundamental theorem about the relationship between the \textbf{R} and \textbf{RE} languages:

If \( L \) is a language, then \( L \in \textbf{R} \) if and only if \( L \in \textbf{RE} \) and \( \bar{L} \in \textbf{RE} \).

This theorem has a beautiful intuition: It says that a language \( L \) is decidable (\( L \in \textbf{R} \)) precisely if, for every string in the language, it’s possible to prove it’s in the language (\( L \in \textbf{RE} \)) and, simultaneously, for every string not in the language, it’s possible to prove it’s not in the language (\( \bar{L} \in \textbf{RE} \)). For this exercise, we’ll look at one of the two directions of this theorem.

Let \( L \) be a language where \( L \in \textbf{RE} \) and \( \bar{L} \in \textbf{RE} \). This means that there’s a recognizer \( M_{\text{yes}} \) for \( L \) and a recognizer \( M_{\text{no}} \) for \( \bar{L} \). In software, you could imagine that \( M_{\text{yes}} \) and \( M_{\text{no}} \) correspond to functions with these signatures:

\[
\text{check}_\text{in}_L(w: \text{str}) \rightarrow \text{bool} \\
\text{check}_\text{in}_L_\bar{L}(w: \text{str}) \rightarrow \text{bool}
\]

Since these functions represent recognizers, they have the following properties:

- For \( w \in \Sigma^* \), \( w \in L \) if and only if \( \text{check}_\text{in}_L(w) \) returns \text{True}.
- For \( w \in \Sigma^* \), \( w \in \bar{L} \) if and only if \( \text{check}_\text{in}_L_\bar{L}(w) \) returns \text{True}.

And we can show that \( L \in \textbf{R} \) by writing pseudocode for a function

\[
is\text{\_in}_L(w: \text{str}) \rightarrow \text{bool}
\]

that takes as input a string \( w \), then returns \text{True} if \( w \in L \) and returns \text{False} if \( w \notin L \):

```python
def is\_in\_L(w: str) -> bool:
    if check\_in\_L(w):
        return True
    if check\_in\_L\_bar(w):
        return False
```

This has the right idea, but it won’t work! Explain what’s wrong and correct the pseudocode. (It’s just pseudocode, so you can make up appropriate notation as long as it’s clear what you mean.)