Regular Expressions

21 September 2021
Where are we?
A language $L$ is a *regular language* if there is a DFA $D$ such that $L(D) = L$. 
THEOREM. The following are equivalent:

$L$ is a regular language.

There is a DFA for $L$.

There is an NFA for $L$. 
If $w \in \Sigma^*$ and $x \in \Sigma^*$, then $wx$ is the concatenation of $w$ and $x$.

If $L_1$ and $L_2$ are languages over $\Sigma$, the concatenation of $L_1$ and $L_2$ is the language $L_1L_2$, defined as

$$L_1L_2 = \{wx \mid w \in L_1 \text{ and } x \in L_2\}.$$  

For example, if $L_1 = \{a, \text{ba}, \text{bb}\}$ and $L_2 = \{\text{aa}, \text{bb}\}$, then

$$L_1L_2 = \{\text{aaa}, \text{abb}, \text{baaa}, \text{babb}, \text{bbaa}, \text{bbbb}\}$$
Lots of concatenation

Consider the language $L = \{aa, b\}$

$LL$ is the set of strings formed by concatenating pairs of strings in $L$:

$\{aaaa, aab, baa, bb\}$

$LLL$ is the set of strings formed by concatenating triples of strings in $L$:

$\{aaaaaa, aaaaab, aabaa, aabb, baaaa, baab, bbaa, bbb\}$

$LLLL$ is the set of strings formed by concatenating quadruples of strings in $L$…
We can define what it means to “exponentiate” a language as follows:

\[ L^0 = \{ \epsilon \} \]

**Base case:** Any string formed by concatenating zero strings together is just the empty string.

\[ L^{n+1} = L \cdot L^n \]

**Recursive case:** Concatenating \( n + 1 \) strings together works by concatenating \( n \) strings, then concatenating one more.
An important operation on languages is the Kleene closure, which is defined as

$$L^* = \{w \in \Sigma^* | \exists n \in \mathbb{N}_0 . w \in L^n\}.$$ 

That is, a word is in $L^*$ iff it’s in

the language $L^0$ or
the language $L^1$ or
the language $L^2$ or ...
$L^*$ consists of all the possible ways of concatenating zero or more strings in $L$.

If $L = \{a, bb\}$, then $L^* = \{ \varepsilon, a, bb, aa, abb, bba, bbbb, aaa, aabb, abba, abbbb, bbba, bbbba, bbbbbbb, ... \}$
Last class, we saw that the class of regular languages \((\text{REG})\) is \textit{closed} under the following operations:

- Complement
- Union
- Intersection
- Concatenation
- Kleene star
That is, if $L_1$ and $L_2$ are regular languages over an alphabet $\Sigma$, then so are the following languages:

- **Complement** \( \overline{L_1} \)
- **Union** \( L_1 \cup L_2 \)
- **Intersection** \( L_1 \cap L_2 \)
- **Concatenation** \( L_1 L_2 \)
- **Kleene star** \( L_1^* \)
Another view of regular languages
We’ve seen we can show a language is regular by constructing a DFA for it or constructing an NFA for it.

We can also show a language is regular by using closure properties to build it out of other regular languages.
This is a bottom-up approach to the regular languages:

Start with a small set of simple languages we know to be regular.
Use closure properties to combine these to form more elaborate languages.

Photograph by
Benjamin D. Esham
**Regular expressions** provide a concise notation for describing this way of building regular languages out of simpler pieces.

They’re use just about everywhere:

- They’re built into JavaScript and used for data validation.
- They’re used in the Unix `grep` tool to search for strings and `flex` to build compilers.
- They’re used to clean and scrape data for large-scale analysis projects.
**DEFINITION**  
*R* is a *regular expression* if *R* is

1. **α** for some $\alpha \in \Sigma$
2. ε
3. $\emptyset$

4. $(R_1 \cup R_2)$, where $R_1$ and $R_2$ are regular expressions
5. $(R_1 \circ R_2)$, where $R_1$ and $R_2$ are regular expressions
6. $(R_1*)$, where $R_1$ is a regular expression
**Order of operations**

We can omit parentheses to make regular expressions more compact, but this makes them ambiguous unless we define precedence:

1. Parentheses \((R)\)
2. Kleene star \(R^*\)
3. Concatenation \(R_1 \circ R_2\) or \(R_1 R_2\)
4. Union \(R_1 \cup R_2\)
Examples

$L(\text{hi}) = \{\text{hi}\}$

$L(\text{hi} \cup \text{heyy}^*) = \{\text{hi, hey, heyy, heyyy, ...}\}$
Examples

\[ L( ((0(0 \cup 1))^*) ) \]

= \[ 0 [0 \text{ or } 1] 0 [0 \text{ or } 1] 0 [0 \text{ or } 1] \ldots \]

The set of strings of 0s and 1s, of even length, such that every odd position has a 0.
The *language of a regular expression* is the language described by that regular expression. Formally,

\[ L(\varepsilon) = \{\varepsilon\} \]

\[ L(\emptyset) = \emptyset \]

\[ L(a) = \{a\} \]

\[ L(R_1 R_2) = L(R_1) L(R_2) \]

\[ L(R_1 \cup R_2) = L(R_1) \cup L(R_2) \]

\[ L(R^*) = L(R)^* \]

\[ L((R)) = L(R) \]
Designing regular expressions
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains } aa \text{ as a substring}\}$
Designing regular expressions

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$(a \cup b)^*aa(a \cup b)^*$
Designing regular expressions

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$$(a \cup b)^*aa(a \cup b)^*$$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | w$ contains $aa$ as a substring$\}$

$$(a \cup b)^* aa (a \cup b)^*$$

bbabbbbaabab

aaaa

bbbbbabbbbaabbbbbb
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | w \text{ contains } aa \text{ as a substring}\}$

$$(a \cup b)^*aa(a \cup b)^*$$

$$\text{bbabbbbaabab}$$

$$\text{aaaa}$$

$$\text{bbbbbbabbbbbabbbbb}$$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains } aa \text{ as a substring}\}$

$\Sigma^*aa\Sigma^*$  \textit{A convenient shorthand}

$bbabbbbaabab$

$aaaa$

$bbbbbabbbbbaabbbbb$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$

Recall: $|w|$ denotes the length of string $w$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | |w| = 4\}$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | |w| = 4\}$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | |w| = 4\}$

$\Sigma^4$

aaaa
baba
bbbb
baaa
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$

Another shorthand

$\Sigma^4$

aaaa
baba
bbbb
baaa
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$

Here are some candidate regular expressions for $L$. Which are correct?

$\Sigma^*a\Sigma^*$

$b^*a^*b^*u^*b^*$

$b^*(a^*\epsilon)b^*$

$b^*a^*b^*u^*b^*$

$b^*(a^*\epsilon\epsilon)b^*$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* | w$ contains at most one $a\}$

$b^*(a \lor \varepsilon) b^*$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$

$b^* (a \cup \varepsilon) b^*$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$

$$b^*(a \cup \varepsilon) b^*$$

- bbbbabbb
- bbbbbbb
- abbb
- a
**Designing regular expressions**

Let \( \Sigma = \{a, b\} \)

Let \( L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\} \)

\[
\text{b* (a∪ε) b*}
\]

- \( bbbbbabbb \)
- \( bbbbbbb \)
- \( abbb \)
- \( a \)
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$

- $b^*a?b^*$
- $bbbbabbb$
- $bbbbbb$
- $abbb$
- $a$
Designing regular expressions

Let $\Sigma = \{a, b\}$

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$

Another shorthand
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
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$$aa^*$$

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\[
\text{aa}^*
\]

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

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$$aa^*(.aa^*)^*$$

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$$aa^*(.aa^*)*$$

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$$aa*(.aa*)*@$$

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$$aa*(.aa*)*@$$

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa*(.aa*)*@aa*.aa*$$

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^* (.aa^*)^* @ aa^*. aa^*$$

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa*(.aa*)*@aa*.aa*(.aa*)*$$

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

```
aa*(.aa*)*@aa*.aa*(.aa*)*
```

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matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+ (.aa^*)*@aa^*.aa^* (.aa^*)*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

You guessed it – another shorthand

$$a^+ (.aa^*)*@aa^*.aa^*(.aa^*)*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+ (.a^+)*@a^+.a^+ (.a^+)*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+(\cdot a^+)*a^+(\cdot a^+)^+$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
For comparison

\[ a^+ (a^+) \ast @ a^+ (a^+) \ast \]
Shorthands summary

Σ is a shorthand for “any character in Σ”

$R^n$ is a shorthand for $RR\ldots R$ ($n$ times)

$R?$ is shorthand for $(R\cup \varepsilon)$ – that is, zero or one copies of $R$.

$R^+$ is a shorthand for $RR^*$ – that is, one or more copies of $R$. 
The power of regular expressions
THEOREM If $R$ is a regular expression, then $L(R)$ is regular.
THEOREM  If $R$ is a regular expression, then $L(R)$ is regular.

PROOF IDEA  Use induction!

The atomic regular expressions all represent regular languages.

The combination steps represent closure properties.

So, anything you can make from them must be regular!
In practice, many regex matchers – including `grep` – use an algorithm called *Thompson’s algorithm* to convert regular expressions into equivalent finite automata.

The “Thompson” is computing pioneer Ken Thompson, a co-inventor of Unix.
That ends the first part of the proof of Theorem 1.54, giving the easier direction of the if and only if condition. Before going on to the other direction, let's consider some examples whereby we use this procedure to convert a regular expression to an NFA.

**Example 1.56**

We convert the regular expression $(ab \cup a)^*$ to an NFA in a sequence of stages.

We build up from the smallest subexpressions to larger subexpressions until we have an NFA for the original expression, as shown in the following diagram.

Note that this procedure generally doesn't give the NFA with the fewest states. In this example, the procedure gives an NFA with eight states, but the smallest equivalent NFA has only two states. Can you find it?

**Solution from Sipser**
THEOREM  If $L$ is a regular language, then there is a regular expression for $L$.

PROOF IDEA  Show how to convert an arbitrary NFA into a regular expression.
Generalizing NFAs
Generalizing NFAs
Generalizing NFAs

These are all regular expressions!
Generalizing NFAs

Note: NFAs aren’t allowed to have transitions like these. This is just a thought experiment.
Generalizing NFAs

- Start state: $q_0$
- Transitions:
  - $q_0 \xrightarrow{ab \cup b} q_1$
  - $q_1 \xrightarrow{ab^*} q_0$
  - $q_0 \xrightarrow{a} q_2$
  - $q_2 \xrightarrow{a^*b?a^*} q_3$
  - $q_3 \xrightarrow{ab^*} q_2$
Generalizing NFAs

\begin{align*}
q_0 & \xrightarrow{a} q_2 & q_2 & \xrightarrow{a^*b^*a^*} q_3 \\
q_0 & \xrightarrow{ab \cup b} q_1 & q_1 & \xrightarrow{ab^*} q_3
\end{align*}

Input sequence: \texttt{a a a b a a b b b b}
Generalizing NFAs

\[
\begin{align*}
q_0 &\xrightarrow{a} q_2 &\xrightarrow{ab^*} q_3 \\
q_2 &\xrightarrow{a*?a^*} q_3 \\
q_0 &\xrightarrow{ab \cup b} q_1 \\
\end{align*}
\]
Generalizing NFAs
Generalizing NFAs
Generalizing NFAs

\[ \text{start} \rightarrow q_0 \xrightarrow{ab \cup b} q_1 \]

\[ \xrightarrow{a} q_2 \xrightarrow{a^*?a^*} q_3 \xrightarrow{ab^*} \]

\[ a \ a \ a \ b \ a \ a \ b \ b \ b \]
Generalizing NFAs

Start

$a$ $a b ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*$ $a b^* ? a^*
Generalizing NFAs
Generalizing NFAs
Generalizing NFAs

- Start state $q_0$
  - Transitions:
    - $a$: $q_0 \rightarrow q_2$
    - $ab$: $q_0 \rightarrow q_1$
    - $ab^*$: $q_2 \rightarrow q_3$
  - Final state $q_1$

- Alphabet: $a, b$
Key idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Key idea 2: If we can convert an NFA into a generalized NFA that looks like this,

then we can easily read off a regular expression for the original NFA.
From GNFAs to regular expressions

\( R_{00}, R_{01}, R_{11}, \) and \( R_{10} \) are variables for arbitrary regular expressions.
From GNFAs to regular expressions

Can we get a clean regular expression from this NFA?
From GNFAs to regular expressions

Key idea 3: Transform a GNFA so it looks like this:

```
start
\[ q_0 \] -> \[ R_{00} \] \[ R_{01} \] \[ R_{10} \] \[ R_{11} \] \[ q_1 \]
```

\[ \text{some-regex} \]
From GNFAs to regular expressions

First add new start and accept states
From GNFAs to regular expressions

First add new start and accept states
From GNFA\text{s} to regular expressions

First add new start
and accept states
From GNFAs to regular expressions

Could we eliminate this state from the GNFA?
From GNFAs to regular expressions
From GNFAs to regular expressions

We can use concatenation and Kleene closure to skip this state.
From GNFAs to regular expressions

\[ q_s \xrightarrow{\varepsilon} q_0 \xrightarrow{R_{00}} q_0 \xrightarrow{R_{01}} q_1 \xrightarrow{R_{11}} q_f \]
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions

We can use union to combine these transitions.
Could we eliminate this state from the GNFA?
From GNFA to regular expressions

Could we eliminate this state from the GNFA?
From GNFAs to regular expressions

\[ R_{00}^* R_{01} \]

\[ R_{11} \cup R_{10} R_{00}^* R_{01} \]
From GNFAs to regular expressions

What should we put on this transition?

\[ R_{00}^* R_{01} \]

\[ R_{11} \cup R_{10} R_{00}^* R_{01} \]
From GNFAs to regular expressions

\[ R_{00}^* R_{01} \left( R_{11} \cup R_{10} R_{00}^* R_{01} \right)^* \varepsilon \]
From GNFA's to regular expressions

\[ R_{00} R_{01} (R_{11} \cup R_{10} R_{00} R_{01})^* \varepsilon \]
From GNFAs to regular expressions

\[ R_{00}^* R_{01} (R_{11} \cup R_{10} R_{00}^* R_{01})^* \varepsilon \]
From GNFAs to regular expressions

\[ R_{00} R_{01} (R_{11} \cup R_{10} R_{00} R_{01})^* \]
From GNFAs to regular expressions

\[ R_{00}^* R_{01} (R_{11} \cup R_{10} R_{00}^* R_{01})^* \]
From GNFAs to regular expressions

Before:

After:

\[ R_{00}^* R_{01} (R_{11} \cup R_{10} R_{00}^* R_{01})^* \]
The state-elimination algorithm

1 Start with an NFA $M$ for the language $L$, which we’ll use as a generalized NFA (GNFA).

2 Add a new start state $q_s$ and accept state $q_f$ to $M$.
   - Add an $\epsilon$-transition from $q_s$ to the old start state of $M$.
   - Add $\epsilon$-transitions from each accepting state of $M$ to $q_f$, then mark them as not accepting.

3 Repeatedly remove states other than $q_s$ and $q_f$ from the NFA by “shortcutting” them until only $q_s$ and $q_f$ remain.

4 The transition from $q_s$ to $q_f$ is now a regular expression equivalent to the original NFA.
Eliminating a state

To eliminate a state \( q_{\text{rip}} \) from the automaton, do the following for each pair of states \( q_i \) and \( q_j \), where there’s a transition from \( q_i \) into \( q_{\text{rip}} \) and a transition from \( q_{\text{rip}} \) into \( q_j \):

1. Let \( R_{\text{in}} \) be the regex. on the transition from \( q_i \) to \( q_{\text{rip}} \).
2. Let \( R_{\text{out}} \) be the regex. on the transition from \( q_{\text{rip}} \) to \( q_j \).
3. If there is a regular expression \( R_{\text{stay}} \) on a transition from \( q_{\text{rip}} \) to itself,
   - Add a new transition from \( q_i \) to \( q_j \) labeled \( ((R_{\text{in}}) (R_{\text{stay}})^* (R_{\text{out}})) \).
4. Otherwise,
   - Add a new transition from \( q_i \) to \( q_j \) labeled \( ((R_{\text{in}}) (R_{\text{out}})) \).

If a pair of states has multiple transitions between them labeled \( R_1, R_2, \ldots, R_k \), replace them with a single transition labeled \( R_1 \cup R_2 \cup \cdots \cup R_k \).
Our transformations

- Direct conversion
- Subset construction
- State elimination
- Thompson’s algorithm
- Regex
- DFA
- NFA
The following are all equivalent:

$L$ is a regular language.

There is a DFA $D$ such that $L(D) = L$.

There is an NFA $N$ such that $L(N) = L$.

There is a regular expression $R$ such that $L(R) = L$. 
Why this matters

The equivalence of regular expressions and finite automata has *practical* relevance.

Tools like `grep` and `flex` that use regular expressions capture all the power available via DFAs and NFAs.

This is also hugely significant *theoretically*:

The regular languages can be assembled “from scratch” using a small number of operations!
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