Regular Expressions

20 September 2022
Where are we?
A language \( L \) is a \textit{regular language} if there is a DFA \( D \) such that \( L(D) = L \).
THEOREM  The following are equivalent:

\( L \) is a regular language.

There is a DFA for \( L \).

There is an NFA for \( L \).
If $w \in \Sigma^*$ and $x \in \Sigma^*$, then $wx$ is the *concatenation* of $w$ and $x$.

If $L_1$ and $L_2$ are languages over $\Sigma$, the *concatenation* of $L_1$ and $L_2$ is the language $L_1L_2$, defined as

$$L_1L_2 = \{wx \mid w \in L_1 \text{ and } x \in L_2\}.$$

For example, if $L_1 = \{a, \text{ba}, \text{bb}\}$ and $L_2 = \{\text{aa}, \text{bb}\}$, then

$$L_1L_2 = \{\text{aaa}, \text{abb}, \text{baaa}, \text{babb}, \text{bbaa}, \text{bbbb}\}$$
Lots of concatenation

Consider the language $L = \{\text{aa, b}\}$

$LL$ is the set of strings formed by concatenating pairs of strings in $L$:

$\{\text{aaaa, aab, baa, bb}\}$

$LLL$ is the set of strings formed by concatenating triples of strings in $L$:

$\{\text{aaaaaa, aaaaab, aabaa, aabb, baaaa, baab, bbaa, bbb}\}$

$LLLLL$ is the set of strings formed by concatenating quadruples of strings in $L$...
We can define what it means to “exponentiate” a language as follows:

$L^0 = \{\varepsilon\}$

**Base case:** Any string formed by concatenating zero strings together is just the empty string.

$L^{n+1} = L \cdot L^n$

**Recursive case:** Concatenating $n + 1$ strings together works by concatenating $n$ strings, then concatenating one more.
We can define what it means to “exponentiate” a language as follows:

$$L^0 = \{\varepsilon\}$$

**Base case:** Any string formed by concatenating zero strings together is just the empty string.

$$L^{n+1} = L \cdot L^n$$

**Recursive case:** Concatenating $n+1$ strings together works by concatenating $n$ strings, then concatenating one more.
[Go back to the end of last class for Kleene closure]
Another view of regular languages
We’ve seen we can show a language is regular by constructing a DFA for it or constructing an NFA for it.

We can also show a language is regular by using closure properties to build it out of other regular languages.
This is a bottom-up approach to the regular languages:

Start with a small set of simple languages we know to be regular.
Use closure properties to combine these to form more elaborate languages.
Regular expressions provide a concise notation for describing this way of building regular languages out of simpler pieces.

They’re use just about everywhere:

- They’re built into JavaScript and used for data validation.
- They’re used in the Unix `grep` tool to search for strings and `flex` to build compilers.
- They’re used to clean and scrape data for large-scale analysis projects.
For the moment, put aside anything you know about writing regular expressions when programming.
Atomic regular expressions

<table>
<thead>
<tr>
<th>Regular expression</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>${\varepsilon}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>${\alpha}$</td>
</tr>
</tbody>
</table>

for any $\alpha \in \Sigma$
## Compound regular expressions

<table>
<thead>
<tr>
<th>Regular expression</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(R_1 \cup R_2)$</td>
<td>$L(R_1) \cup L(R_2)$</td>
</tr>
<tr>
<td>$(R_1 \circ R_2)$</td>
<td>$L(R_1) \circ L(R_2)$</td>
</tr>
<tr>
<td>$(R_1^*)$</td>
<td>$L(R_1)^*$</td>
</tr>
</tbody>
</table>

for any regular expressions $R_1$ and $R_2$
**DEFINITION**  \( R \) is a *regular expression* if \( R \) is

1. \( \alpha \) for some \( \alpha \in \Sigma \)
2. \( \varepsilon \)
3. \( \emptyset \)

**Basis**

- \( (R_1 \cup R_2) \), where \( R_1 \) and \( R_2 \) are regular expressions
- \( (R_1 \circ R_2) \), where \( R_1 \) and \( R_2 \) are regular expressions

**Recursive cases**

- \( (R_1^*) \), where \( R_1 \) is a regular expression
As with strings and languages, we can omit the \( \circ \) and just indicate concatenation by writing two regular expressions next to each other.

\[
ab = a \circ b
\]
Order of operations

We can omit parentheses to make regular expressions more compact, but this makes them ambiguous unless we define precedence:

1. Parentheses \((R)\)
2. Kleene star \(R^*\)
3. Concatenation \(R_1 \circ R_2\) or \(R_1 R_2\)
4. Union \(R_1 \cup R_2\)
ab*cud
ab*cud

a(b*)cud
\(ab^*cud\)

\(a(b^*)cud\)

\((a \circ (b^*))cud\)
ab*cuda

a(b*)cud

(a°(b*))cud

((a°(b*))°c)ud
ab*cud
a(b*)cud
(a°(b*))cud
(((a°(b*))°c)ud)
(((a°(b*))°c)ud)
This is the fully parenthesized version, following our formal, recursive definition of regular expressions.
Examples

$L(oh) = \{oh\}$

$L(ohuawww^*) = \{oh, aww, awww, awwww, awww, \ldots\}$

$L((oua)(huwww^*)) = \ldots$
Examples

\[ L(oh) = \{oh\} \]

\[ L(ohuawww^*) = \{oh, aww, awww, awwww, aww, \ldots\} \]

\[ L((oua) (huww^*)) = \{ \]
  \hspace{1cm}
oh,  
  \hspace{1cm}oww, owww, owww, \ldots,  
  \hspace{1cm}ah,  
  \hspace{1cm}aww, awww, awwww, \ldots
\]
Designing regular expressions
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* | w$ contains $aa$ as a substring$\}$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains } aa \text{ as a substring}\}$.

$$(a \cup b)^* aa (a \cup b)^*$$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w$ contains $aa$ as a substring$\}$. 

$(aUb)^*aa(aUb)^*$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains } aa \text{ as a substring}\}$.

$(a \cup b)^*aa(a \cup b)^*$

bbabbbbaabab

aaaa

bbbbbabbbbaabbbbb
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w$ contains $aa$ as a substring\}.

$$(a \cup b)^*aa(a \cup b)^*$$

bbabbbbaabab

aaaa

bbbbbabbbbbaabbbbbb
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* | w \text{ contains } aa \text{ as a substring}\}$. 

$\Sigma^*aa\Sigma^*$  

bbabbbbaabab  

aaaa  

bbbbbbabbbbaabbbbb
Designing regular expressions

Let \( \Sigma = \{ a, b \} \).

Let \( L = \{ w \in \Sigma^* \mid |w| = 4 \} \).
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 

We write $|w|$ to denote the length of the string $w$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* | |w| = 4\}$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 

$\Sigma\Sigma\Sigma\Sigma$

aaaa
baba
bbbb
baaa
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 

\[\Sigma\Sigma\Sigma\Sigma\]

aaaa
baba
bbbb
baaa
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$.
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$.

Another convenient shorthand!
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid |w| = 4\}$. 

Another convenient shorthand!
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$. 
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$.

Here are some candidate regular expressions for $L$. Which are correct?

- $\Sigma^* a \Sigma^*$
- $b^* a b^* u b^*$
- $b^* (a u \varepsilon) b^*$
- $b^* a^* b u b^*$
- $b^* (a^* u \varepsilon) b^*$
Designing regular expressions

Let \( \Sigma = \{a, b\} \).

Let \( L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\} \).

\[
b^* (a \cup \varepsilon) b^*
\]
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* | w$ contains at most one $a\}$.

$b^*(a \cup \epsilon)b^*$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$.

$b^*(a \cup \varepsilon)b^*$

$\text{bbbbabbb}$

$\text{bbbbbbbb}$

$\text{abbb}$

$\text{a}$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w$ contains at most one $a\}$. 

$$b^* (a \cup \varepsilon) b^*$$

$$b b b b a b b b$$

$$b b b b b b b b$$

$$a b b b$$

$$a$$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$.

\[
b^*a?b^*
\]

$bbbbabbb$

$bbbbbb$

$abbb$

$a$
Designing regular expressions

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ contains at most one } a\}$.

Another convenient shorthand!

$b^*a?b^*$

$bbbabbb$

$bbbbbb$

$abbb$

$a$
A more elaborate design

Let $\Sigma = \{a, \cdot, @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

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```
  aa*
```

mvassar@vassar.edu
matthew.vassar@vassarbrewwery.com
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A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^*$$

mvassar@vassar.edu  
matthew.vassar@vassarbrewery.com  
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^*(.aa^*)^*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^* (\cdot aa^*)^*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^*(.aa^*)*@$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

```
aa* (.aa*)*@ 
```

mvassar@vassar.edu

matthew.vassar@vassarbrewery.com

matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa*(.aa*)*@aa*.aa*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

\[
\texttt{aa*(.aa*)*@aa*.aa*}
\]

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa*(.aa*)*@aa*.aa*(.aa*)*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$aa^*(.aa^*)*@aa^*.aa^*(.aa^*)^*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

\[
a^+ (.a^*)^* @a^* .a^* (.a^*)^*
\]

- mvassar@vassar.edu
- matthew.vassar@vassarbrewery.com
- matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

```
a+(.aa*)*@aa*.aa*(.aa*)*
```

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+ ( . a^+ )^* @ a^+ . a^+ ( . a^+ )^*$$

- mvassar@vassar.edu
- matthew.vassar@vassarbrewery.com
- matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+.(a^+)^*@a^+.a^+(a^+)^*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
A more elaborate design

Let $\Sigma = \{a, ., @\}$, where $a$ represents “any letter”.

Let’s make a regular expression for email addresses.

$$a^+.a^+*a^+.a^+*$$

mvassar@vassar.edu
matthew.vassar@vassarbrewery.com
matt@cs.vassar.edu
For comparison

$a^+ (a^+) * a^+ (a^+) ^+$
Shorthands summary

Σ is a shorthand for “any character in Σ”

$R^n$ is a shorthand for $RR\ldots R$ ($n$ times)

$R?$ is shorthand for $(R∪ε)$ – that is, zero or one copies of $R$.

$R^+$ is a shorthand for $RR^*$ – that is, one or more copies of $R$. 
Optional interlude: Unix regular expressions
UNIX regular expressions

From the beginning (of time), UNIX has used regular expressions in many places, including the `grep` command.

`grep` = global (search for a) regular expression and print

Many UNIX commands use an extended RE notation, but it still expresses only the regular languages.
UNIX RE notation

\([a_1 a_2 \ldots a_n]\) is shorthand for \(a_1 \cup a_2 \cup \cdots \cup a_n\).

Ranges are indicated by first-dash-last and brackets, using ASCII character order, e.g.,

\([a–z]\) = any lowercase letter

\([a–zA–Z]\) = any letter

Dot (\(\cdot\)) = any character (like our shorthand \(\Sigma\))
UNIX RE notation, *continued*

Since characters like brackets, dashes, and dots have special meaning, if you want to match them, you need to quote with backslash (`\`).

Union operator is represented with a bar (`|`)

Includes our `+` shorthand for “one or more”, e.g.,

```
[a-z]+ = one or more lowercase letter
```
Perl, Python, Emacs, …

Include additional extensions, notably character classes like `\b` for word boundary characters, `\w` for word characters, etc.

With each implementation of regular expressions, they become less standard, so what you write for one language or application won’t work in another.
grep

grep lets you search files for text

$ grep bananas foo.txt

Here are some of my favourite grep command line arguments!

- case insensitive
  -i

- Show context for your search.
  -A 3 foo
    will show 3 lines of context after a match

- only show the filenames of the files that matched
  -l

- only show the matching part of the line (not the whole line)
  -o

- search binaries: treat binary data like it's text instead of ignoring it!
  -a

- don't treat the match string as a regex
  -F
    eg $ grep -F ...

- recursive! Search all the files in a directory.
  -r

grep alternatives

ack
ag
ripgrep
(better for searching code!)
The power of regular expressions
Regular Languages
Languages you can build a DFA for
Regular Languages

Languages you can build a DFA for

Languages you can build an NFA for
Regular Languages

Languages you can build a DFA for

Languages you can build an NFA for

Languages you can write a regex for
Regular Languages

Languages you can build a DFA for

Languages you can build an NFA for

Languages you can write a regex for
Languages you can build a DFA for

Languages you can build an NFA for

Languages you can write a regex for

Regular Languages
Languages you can write a regex for

Languages you can build a DFA for

Regular Languages

Languages you can build an NFA for

Languages you can write a regex for
Regular Languages

Languages you can build a DFA for

Languages you can build an NFA for

Languages you can write a regex for
THEOREM  If $R$ is a regular expression, then $L(R)$ is regular.
THEOREM  If $R$ is a regular expression, then $L(R)$ is regular.

PROOF IDEA  Use induction!

The atomic regular expressions all represent regular languages.

The combination steps represent closure properties.

So, anything you can make from them must be regular!
In practice, many regex matchers — including **grep** — use an algorithm called *Thompson’s algorithm* to convert regular expressions into equivalent finite automata.

The “Thompson” is computing pioneer Ken Thompson, a co-inventor of Unix.
Example

\((ab \cup a)^*\)

We convert the regular expression \((ab \cup a)^*\) to an NFA in a sequence of stages. We build up from the smallest subexpressions to larger subexpressions until we have an NFA for the original expression, as shown in the following diagram.

Note that this procedure generally doesn't give the NFA with the fewest states. In this example, the procedure gives an NFA with eight states, but the smallest equivalent NFA has only two states. Can you find it?

Solution from Sipser
Regular Languages

- Languages you can build a DFA for
- Languages you can build an NFA for
- Languages you can write a regex for
Regular Languages

Languages you can build a DFA for
Languages you can build an NFA for
Languages you can write a regex for
THEOREM  If $L$ is a regular language, then there is a regular expression for $L$.

PROOF IDEA  Show how to convert an arbitrary NFA into a regular expression.
Generalizing NFAs
Generalizing NFAs

![NFA Diagram]

- **q₀** (start state)
- **q₁**
- **q₂**
- **q₃**
- **q₄**

Transitions:
- **q₀ → q₁**: on **a**
- **q₀ → q₃**: on **ε**
- **q₃ → q₄**: on **b**
- **q₂ → q₄**: on **b**
- **q₂ → q₃**: on **Σ**
- **q₃ → q₃**: on **Σ**
Generalizing NFAs

These are all regular expressions!
Generalizing NFAs

Note: NFAs aren’t allowed to have transitions like these. This is just a thought experiment.
Generalizing NFAs

\[ \text{start} \quad q_0 \quad \text{ab} \cup b \quad q_1 \]

\[ \text{a} \quad q_2 \quad a^*b?a^* \quad q_3 \]

\[ \text{ab}^* \quad q_1 \quad ab^* \quad q_3 \]
Generalizing NFAs
Generalizing NFAs
Generalizing NFAs

Diagram:
- States: $q_0$, $q_1$, $q_2$, $q_3$
- Transitions:
  - $q_0$ to $q_1$: $ab \cup b$
  - $q_0$ to $q_2$: $a$
  - $q_1$ to $q_3$: $ab^*$
  - $q_2$ to $q_3$: $a^*b?a^*$

Input string:
$aaaabaaaabb$
Generalizing NFAs
Generalizing NFAs

\[
\begin{align*}
q_0 &\xrightarrow{a} q_2 \\
q_2 &\xrightarrow{a*b?a*} q_3 \\
q_0 &\xrightarrow{ab \cup b} q_1 \\
q_2 &\xrightarrow{ab^*} q_3
\end{align*}
\]
Generalizing NFAs
Generalizing NFAs

\[ a \quad a \quad a \quad b \quad a \quad a \quad b \quad b \quad b \]
Generalizing NFAs

- $q_0$: Transition on $a$ to $q_2$.
- $q_1$: Transition on $ab \cup b$ to $q_1$.
- $q_2$: Transition on $a$ to $q_2$.
- $q_3$: Transition on $ab^*?a^*$ to $q_3$.

Transition on $ab \cup b$ from $q_0$ to $q_1$.
Transition on $a$ from $q_0$ to $q_0$.
Transition on $a^*b?a^*$ from $q_2$ to $q_3$.
Transition on $ab^*$ from $q_3$ to $q_3$.

Input sequence: $aaaabaaaabbbb$
Generalizing NFAs
**Key idea 1**: Imagine that we can label transitions in an NFA with arbitrary regular expressions.
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Generalizing NFAs

Is there a simple regular expression for the language of this generalized NFA?
Key idea 2: If we can convert an NFA into a generalized NFA that looks like this,

![Diagram](image)

then we can easily read off a regular expression for the original NFA.
From GNFA\'s to regular expressions

\[ \begin{array}{c}
R_{00}, \ R_{01}, \ R_{11}, \text{ and } R_{10} \text{ are variables for arbitrary regular expressions.}
\end{array} \]
From GNFA to regular expressions

Can we get a clean regular expression from this NFA?
From GNFAs to regular expressions

Key idea 3: Transform a GNFA so it looks like this:
From GNFAs to regular expressions

First add new start and accept states
From GNFAs to regular expressions

First add new start and accept states
From GNFAs to regular expressions

First add new start and accept states
Could we eliminate this state from the GNFA?
From GNFAs to regular expressions
We can use concatenation and Kleene closure to skip this state.
From GNFA$s$ to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFAs to regular expressions
From GNFA's to regular expressions

\[
\begin{align*}
R_{00}R_{01} & \\
R_{11} & \\
R_{10}R_{00}R_{01} & \\
\varepsilon & \\
\end{align*}
\]
From GNFA to regular expressions

\begin{align*}
\text{start} & \rightarrow q_s \\
R_{00}^* R_{01} & \rightarrow q_1 \\
R_{10} R_{00}^* R_{01} & \rightarrow q_f
\end{align*}
From GNFAs to regular expressions

We can use union to combine these transitions.
From GNFAs to regular expressions

Could we eliminate this state from the GNFA?
From GNFAs to regular expressions

Could we eliminate this state from the GNFA?
From GNFAs to regular expressions
From GNFAs to regular expressions

What should we put on this transition?
From GNFAs to regular expressions

\[ R_{00} R_{01} (R_{11} \cup R_{10} R_{00} R_{01})^* \varepsilon \]
From GNFAs to regular expressions

\[ R_{00}R_{01}(R_{11} \cup R_{10}R_{00}R_{01})^*\varepsilon \]
From GNFA's to regular expressions

\( R_{00}^* R_{01} (R_{11} \cup R_{10} R_{00}^* R_{01})^* \varepsilon \)
From GNFAs to regular expressions

\[ R_{00} R_{01} (R_{11} \cup R_{10} R_{00} R_{01})^* \]
From GNFAs to regular expressions
From GNFAs to regular expressions

After:

\[ R_{00}^* R_{01} (R_{11} \cup R_{10} R_{00}^* R_{01})^* \]

Before:
The state-elimination algorithm

1 Start with an NFA $M$ for the language $L$, which we’ll use as a generalized NFA (GNFA).

2 Add a new start state $q_s$ and accept state $q_f$ to $M$.
   
   Add an $\varepsilon$-transition from $q_s$ to the old start state of $M$.
   Add $\varepsilon$-transitions from each accept state of $M$ to $q_f$, then mark them as not accept states.

3 Repeatedly remove states other than $q_s$ and $q_f$ from the NFA by “shortcutting” them until only $q_s$ and $q_f$ remain.

4 The transition from $q_s$ to $q_f$ is now a regular expression equivalent to the original NFA.
To eliminate a state $q_{rip}$ from the automaton, do the following for each pair of states $q_i$ and $q_j$, where there's a transition from $q_i$ into $q_{rip}$ and a transition from $q_{rip}$ into $q_j$:

Let $R_{in}$ be the regex. on the transition from $q_i$ to $q_{rip}$.

Let $R_{out}$ be the regex. on the transition from $q_{rip}$ to $q_j$.

If there is a regular expression $R_{stay}$ on a transition from $q_{rip}$ to itself,

Add a new transition from $q_i$ to $q_j$ labeled ($(R_{in})(R_{stay})^*(R_{out}))$.

Otherwise,

Add a new transition from $q_i$ to $q_j$ labeled ($(R_{in})(R_{out}))$.

If a pair of states has multiple transitions between them labeled $R_1$, $R_2$, …, $R_k$, replace them with a single transition labeled $R_1 \cup R_2 \cup \cdots \cup R_k$. 
Our transformations

- Direct conversion
- Subset construction
- State elimination
- Thompson’s algorithm
The following are all equivalent:

$L$ is a regular language.

There is a DFA $D$ such that $L(D) = L$.

There is an NFA $N$ such that $L(N) = L$.

There is a regular expression $R$ such that $L(R) = L$. 
Why this matters

The equivalence of regular expressions and finite automata has *practical* relevance.

Tools like *grep* and *flex* that use regular expressions capture all the power available via DFAs and NFAs.

This is also hugely significant *theoretically*:

The regular languages can be assembled “from scratch” using a small number of operations!
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