CMPU 240 · Theory of Computation

Context-free Grammars and Context-free Languages

5 March 2026

Exam 1

Graded soon (hopefully today); solutions posted on Ed.

Assignment 5

Out now

Where are we?

Describing languages

We've seen two models for the regular languages:

Finite automata recognize strings in the language.

They perform a computation to determine whether a specific string is in the language.

Regular expressions match strings in the language.

They describe the general shape of all strings in the language.

The *Pumping Lemma for Regular Languages* let us establish limits on what languages are regular.

To describe more complex languages, we need a more powerful formalism – context-free grammars.

The creation of a thousand forests is in one acorn.

Ralph Waldo Emerson

The creation of a *potentially infinite number of strings* is in one *grammar*.

Ralph Waldo Emerson,
theoretician

A *context-free grammar* (CFG) describes a language by recursively describing the structure of the strings in the language.

Context-free:

$A \rightarrow$ `foo`

Context-sensitive:

`bar` A `baz` $\rightarrow$ `bar foo baz`

We won't use these!

If G is a CFG with alphabet Σ and start symbol S, then the language of G is the set

$$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow}_G w\}$$

That is, $L(G)$ is the set of strings derivable from the start symbol using the production rules in grammar G.

Note that w must be in Σ^*, the set of strings made up only of terminals. Strings with variables aren't in the language!

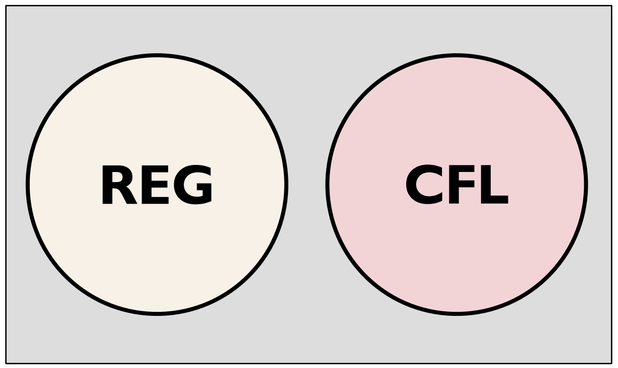
A language L is called a *context-free language* (CFL) if there is a CFG G such that $L = L(G)$.

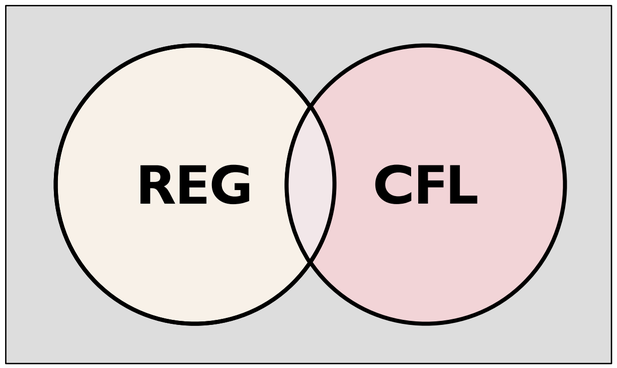
How are context-free and regular languages related?

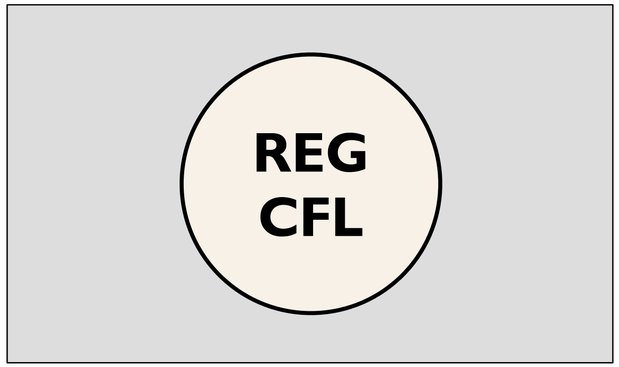
How do we design context-free grammars for context-free languages?

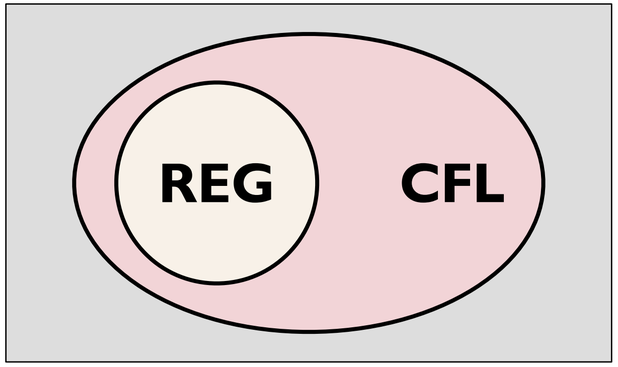
Regular and context-free languages

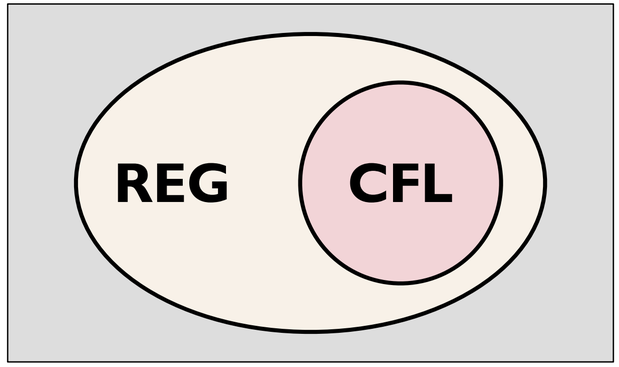
Possible relations











Consider the CFG G:

$S \rightarrow a^*b$

L(G) = …?

Beware!

Consider the CFG G:

$$S \to \mathtt{a}^*\mathtt{b}$$

$L(G) \neq \{\mathtt{a}^n\mathtt{b} \mid n \in \mathbb{N}_0\}$.

$L(G) = \{\mathtt{a}^*\mathtt{b}\}$. It contains exactly one string, $\mathtt{a}^*\mathtt{b}$.

Context-free grammars just consist of production rules with a concatenation of terminals and variables.

They don't have the regular expression operators Kleene star, union, or parenthesized expressions.

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But we can convert regular expressions to CFGs!

THEOREM Every regular language is context-free.

PROOF IDEA Show how to convert an arbitrary regular expression into a context-free grammar for the same language.

In regular expressions, we could write * for zero-or-more repetitions, e.g.,

```
a*b
```

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$S \rightarrow \texttt{a*b}$

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$S \rightarrow \mathtt{a}^*\mathtt{b}$

$A \rightarrow A\mathtt{a} \mid \varepsilon$

In regular expressions, we could write * for zero-or-more repetitions, e.g.,

$S \rightarrow \texttt{a*b}$

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In regular expressions, we could write * for zero-or-more repetitions, e.g.,

$S \rightarrow Ab$

$A \rightarrow Aa \mid \varepsilon$

In regular expressions, we could write parentheses to group expressions and `u` for alternatives, e.g.,

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a(b u c*)
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$S \rightarrow$ `a(b u c*)`

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$S \rightarrow \mathtt{a(b\ u\ c^*)}$

$X \rightarrow \mathtt{b} \mid \mathtt{c^*}$

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$S \rightarrow$ `a(b u c*)`

$X \rightarrow$ `b` | `c*`

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$S \rightarrow aX$

$X \rightarrow b \mid c^*$

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$S \rightarrow \mathbf{a}X$

$X \rightarrow b \mid \mathbf{c}^*$

$C \rightarrow Cc \mid \varepsilon$

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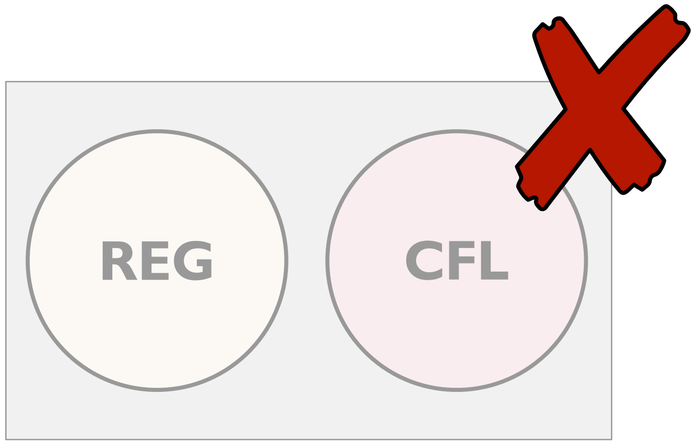
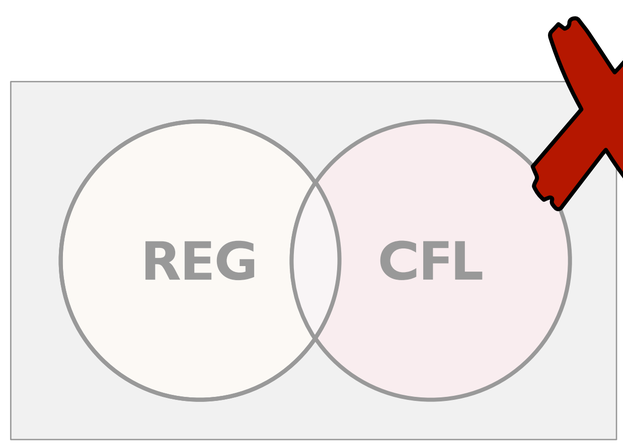
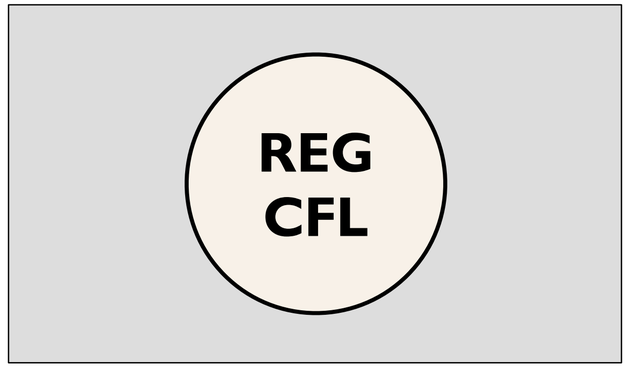
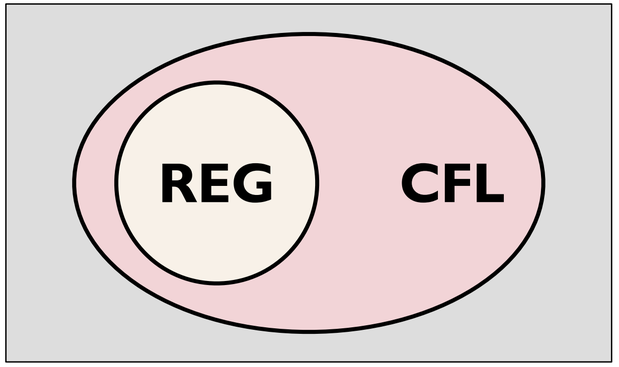
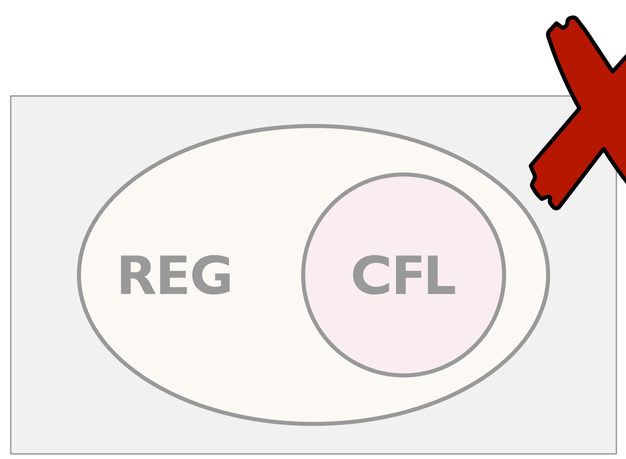
Possible relations











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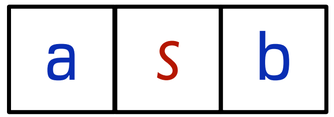

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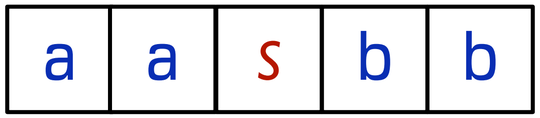

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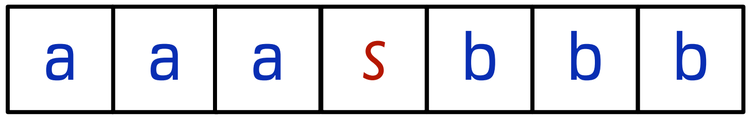

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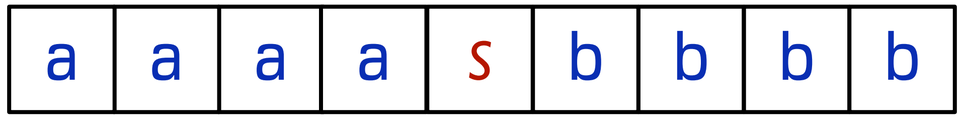

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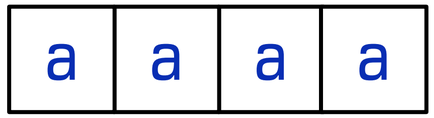

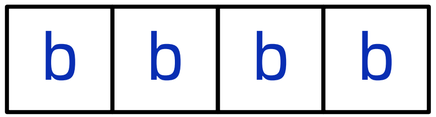

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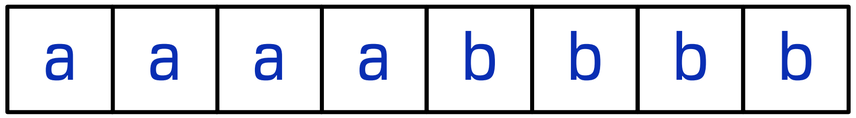

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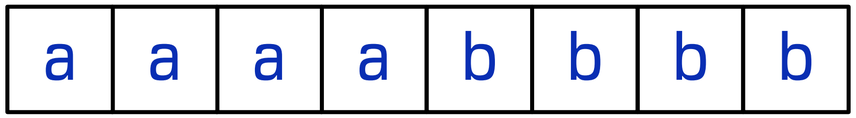
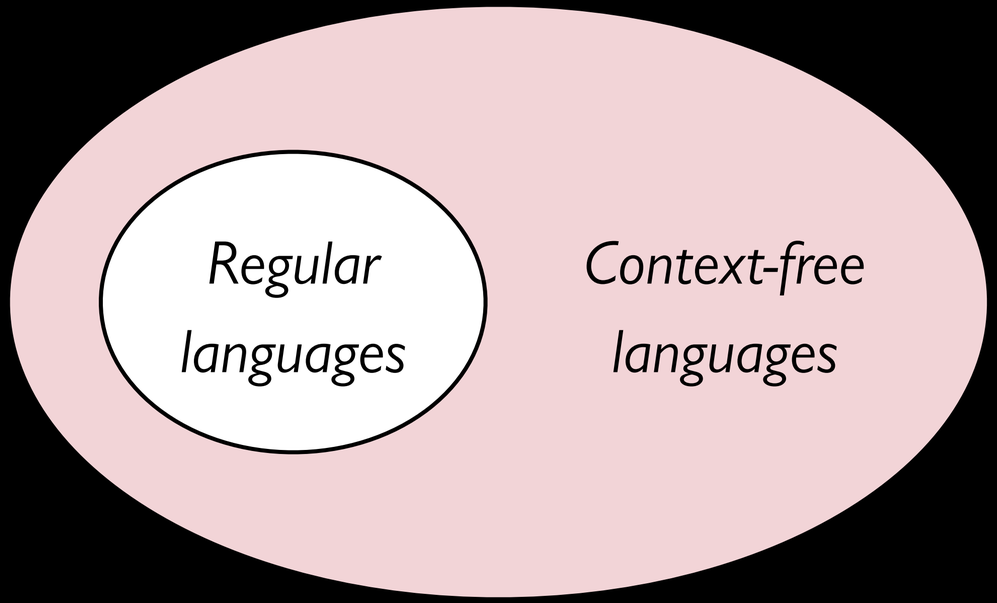


$$L(G) = \{a^n b^n \mid n \in \mathbb{N}_0\}$$



All languages

Why do CFGs have more power than regular expressions?

Intuition: Derivations of strings have unbounded “memory”.

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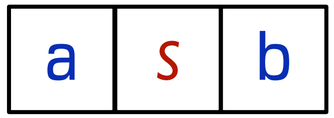

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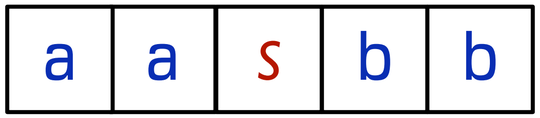

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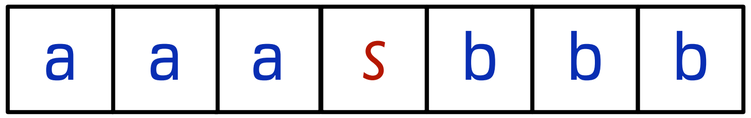

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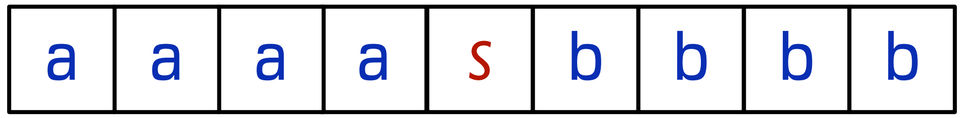

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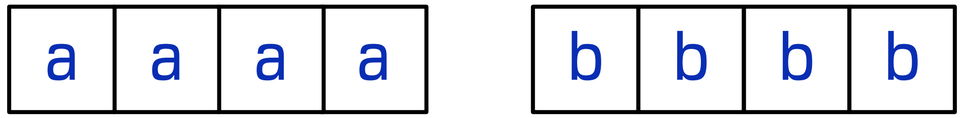

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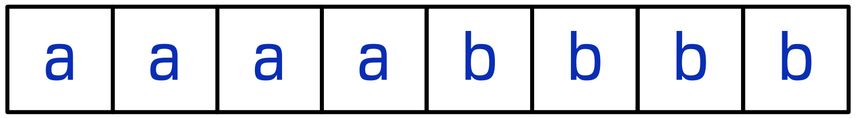

Designing CFGs

Like writing code, designing a grammar is a craft.

Helpful strategies:

Think recursively.

Build bigger structures from smaller ones.

Have a construction plan.

Know what order you'll build up the string.

Store information in variables:

Have each variable correspond to some useful piece of information.

Example 1

Let $\Sigma = \{a, b\}$.

Let $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$.

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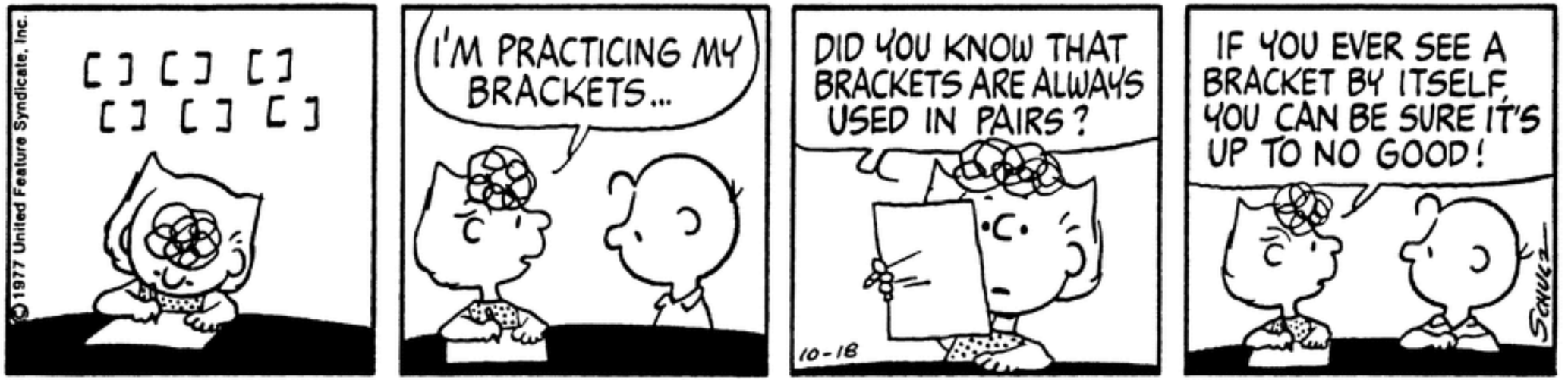
$S \rightarrow \varepsilon \mid a \mid b$

$S \rightarrow aSa \mid bSb$

Example 2

Let $\Sigma = \{$[,]$\}$.

Let $L = \{w \in \Sigma^* \mid w$ is a string of balanced brackets$\}$.



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Let $\Sigma = \{$`[`, `]`$\}$.

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Some sample strings in L:

- ε
- `[[[]]]`
- `[][]`
- `[[]][]`
- `[[][]][[]]`

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Let's think about this recursively:

Base case: The empty string is a string of balanced brackets.

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`[[][[]][[]][[]][[]]]`

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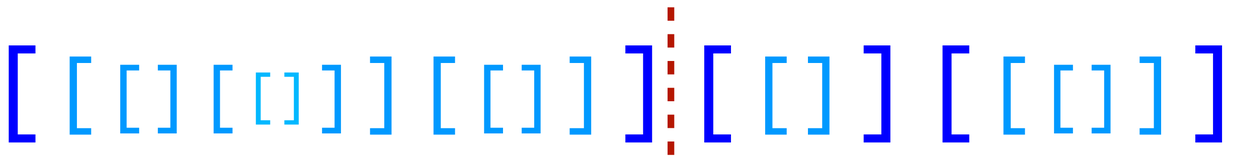

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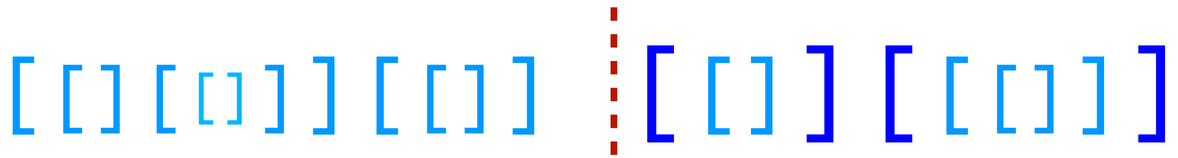

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Recursive case: Look at the closing bracket that matches the first open bracket. Removing the first bracket and the matching bracket forms two new strings of balanced brackets.

$$S \rightarrow \varepsilon \mid SS \mid [S]$$

Designing CFGs

Let $\Sigma = \{a, b\}$ and let $L = \{w \in \Sigma^* \mid n_a(w) = n_b(w)\}$

How many of the following CFGs have language L?

$S \to aSb \mid bSa \mid \varepsilon$

$S \to abS \mid baS \mid \varepsilon$

$S \to abSba \mid baSab \mid \varepsilon$

$S \to SbaS \mid SabS \mid \varepsilon$

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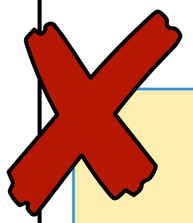

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$S \to abSba \mid baSab \mid \varepsilon$

$S \to SbaS \mid SabS \mid \varepsilon$

Designing CFGs

Let $\Sigma = \{a, b\}$ and let $L = \{w \in \Sigma^* \mid n_a(w) = n_b(w)\}$

How many of the following CFGs have language L?

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Designing CFGs: Caveat 1

You need to make sure your grammar

- can generate *all* of the strings in the language
- can *never* generate a string outside the language

This can be tricky; you'll need to try test cases!

Designing CFGs: Caveat 2

Is this a CFG for the language $\{a^n b^n \mid n \in \mathbb{N}_0\}$?

$$S \to aSb$$

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What is the language of the grammar?

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What strings can you derive?

None!

What is the language of the grammar?

$\varnothing$

When designing CFGs, make sure your recursion actually terminates!

Designing CFGs: Caveat 3

Remember that each variable can be expanded out independently of the others.

E.g., let $\Sigma = \{\mathtt{a}, \mathtt{=}\}$ and $L = \{\mathtt{a}^n\mathtt{=}\mathtt{a}^n \mid n \in \mathbb{N}_0\}$.

$S \to X\mathtt{=}X$

$X \to \mathtt{a}X \mid \varepsilon$

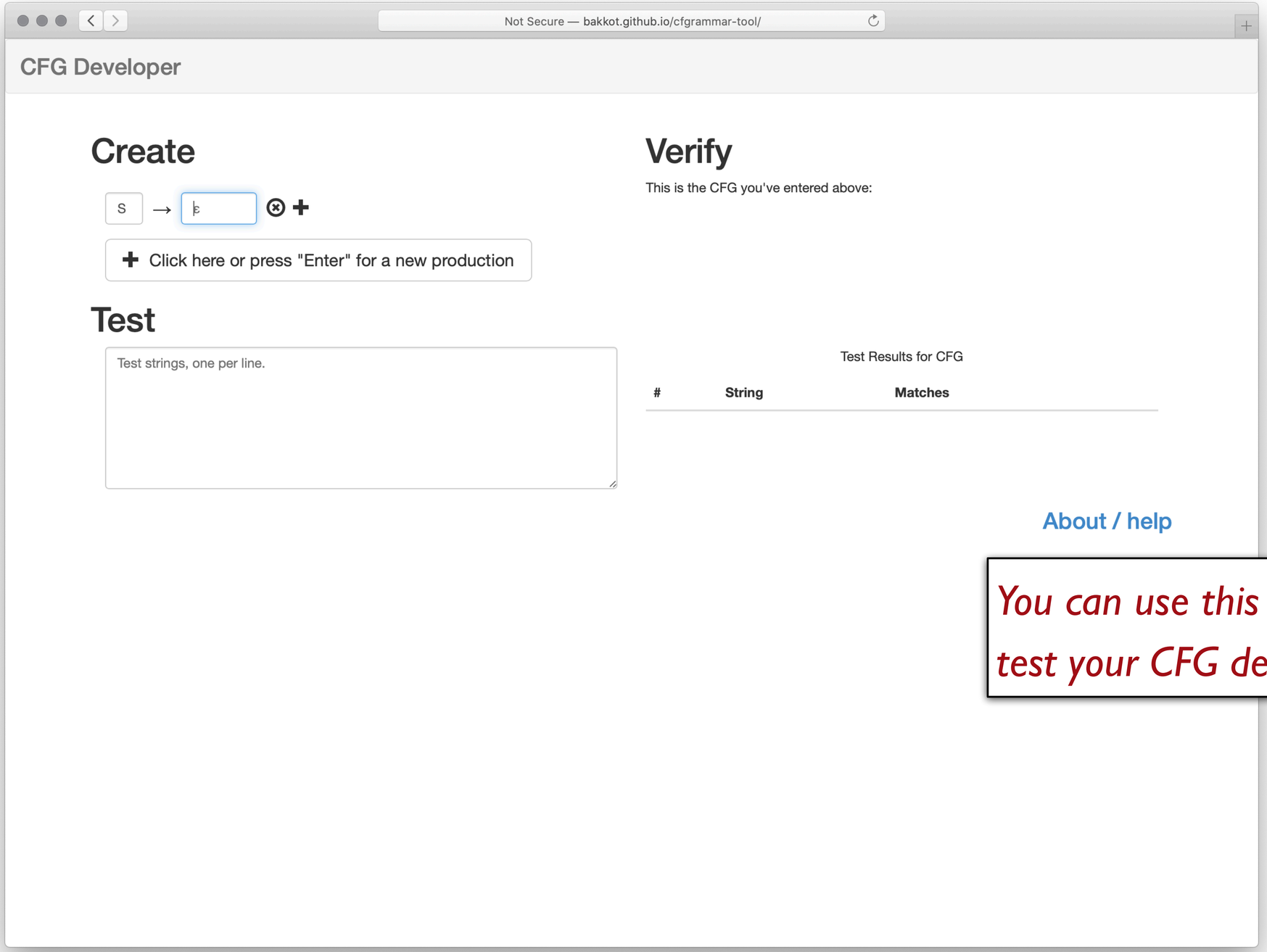
$$
\begin{aligned}
& S \\
\Rightarrow\ & X\mathtt{=}X \\
\Rightarrow\ & \mathtt{a}X\mathtt{=}X \\
\Rightarrow\ & \mathtt{aa}X\mathtt{=}X \\
\Rightarrow\ & \mathtt{aa=}X \\
\Rightarrow\ & \mathtt{aa=a}X \\
\Rightarrow\ & \mathtt{aa=a}
\end{aligned}
$$

Designing CFGs: Caveat 3

We can force a build order, so the string is built from the ends inward, balancing the sides of the equality:

$S \rightarrow aSa \mid =$

$$
\begin{aligned}
& S \\
\Rightarrow\ & aSa \\
\Rightarrow\ & aaSaa \\
\Rightarrow\ & aaaSaaa \\
\Rightarrow\ & aaa=aaa
\end{aligned}
$$


Not Secure — bakkot.github.io/cfgrammar-tool/
CFG Developer
Create
S
→
ε
Click here or press "Enter" for a new production
Test
Test strings, one per line.
Verify
This is the CFG you've entered above:
Test Results for CFG
String Matches
About / help
You can use this tool to test your CFG designs!

Language: Function prototypes

Let Σ = {`void`, `int`, `double`, `name`, `(`, `)`, `,`, `;`}

Let's write a CFG for C-style function prototypes!

Language: Function prototypes

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Let's write a CFG for C-style function prototypes!

Examples:

```
void name(int name, double name);
int name();
int name(double name);
int name(int, int name, int);
void name(void);
```

$$
\begin{aligned}
S &\rightarrow \mathit{Ret}\ \texttt{name (}\mathit{Args}\texttt{);} \\
\mathit{Ret} &\rightarrow \mathit{Type} \mid \texttt{void} \\
\mathit{Type} &\rightarrow \texttt{int} \mid \texttt{double} \\
\mathit{Args} &\rightarrow \varepsilon \mid \texttt{void} \mid \mathit{ArgList} \\
\mathit{ArgList} &\rightarrow \mathit{OneArg} \mid \mathit{ArgList}\texttt{,}\ \mathit{OneArg} \\
\mathit{OneArg} &\rightarrow \mathit{Type} \mid \mathit{Type}\ \texttt{name}
\end{aligned}
$$

Summary of CFG design tips

Look for recursive structures where they exist: they can help guide you toward a solution.

Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.

> Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.

Use different variables to represent different structures.

Acknowledgments

This lecture incorporates material from:

Nancy Ide, Vassar College

Keith Schwarz, Stanford University

Michael Sipser, *Introduction to the Theory of Computation*

Jeffrey Ullman, Stanford University