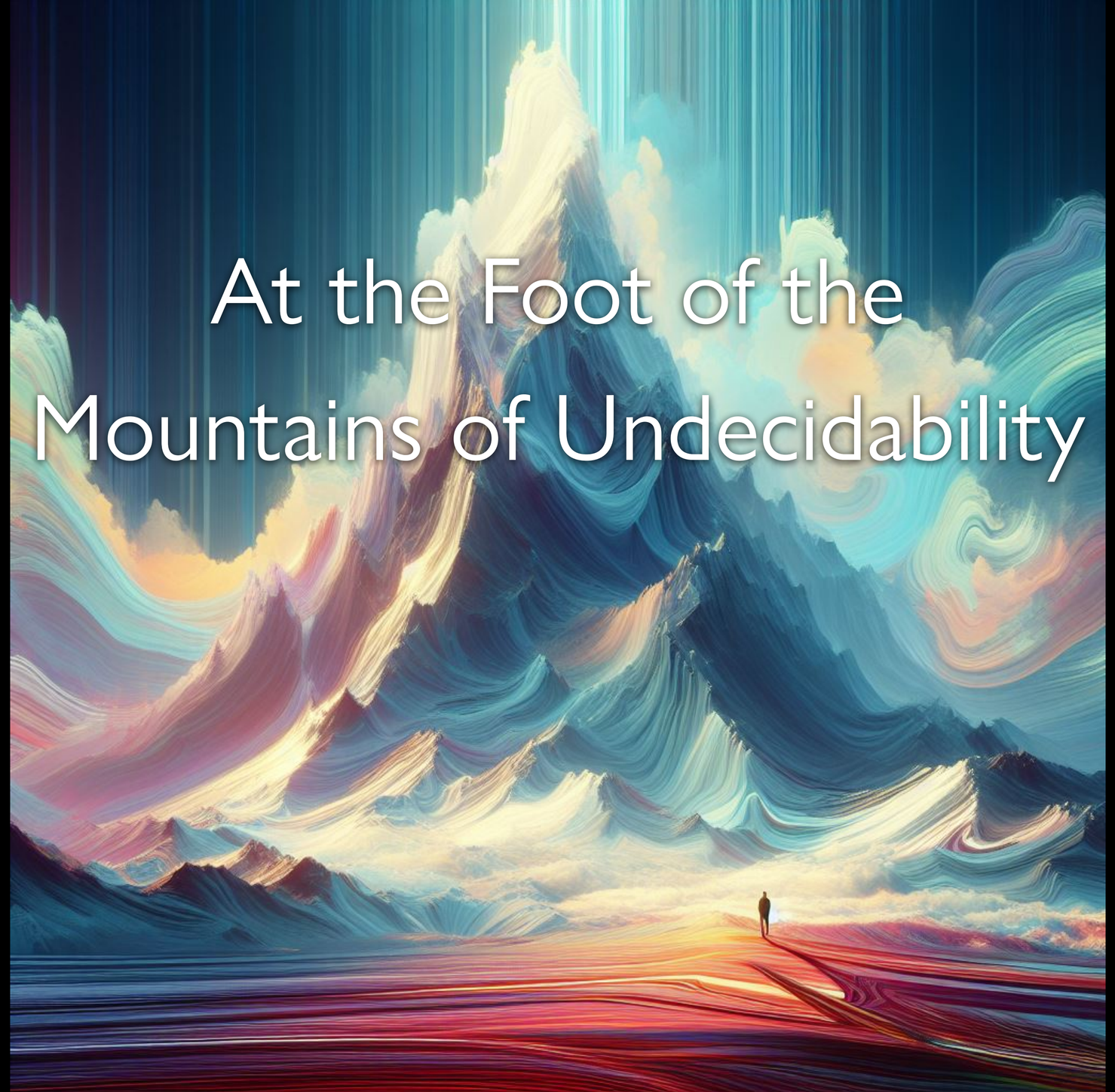
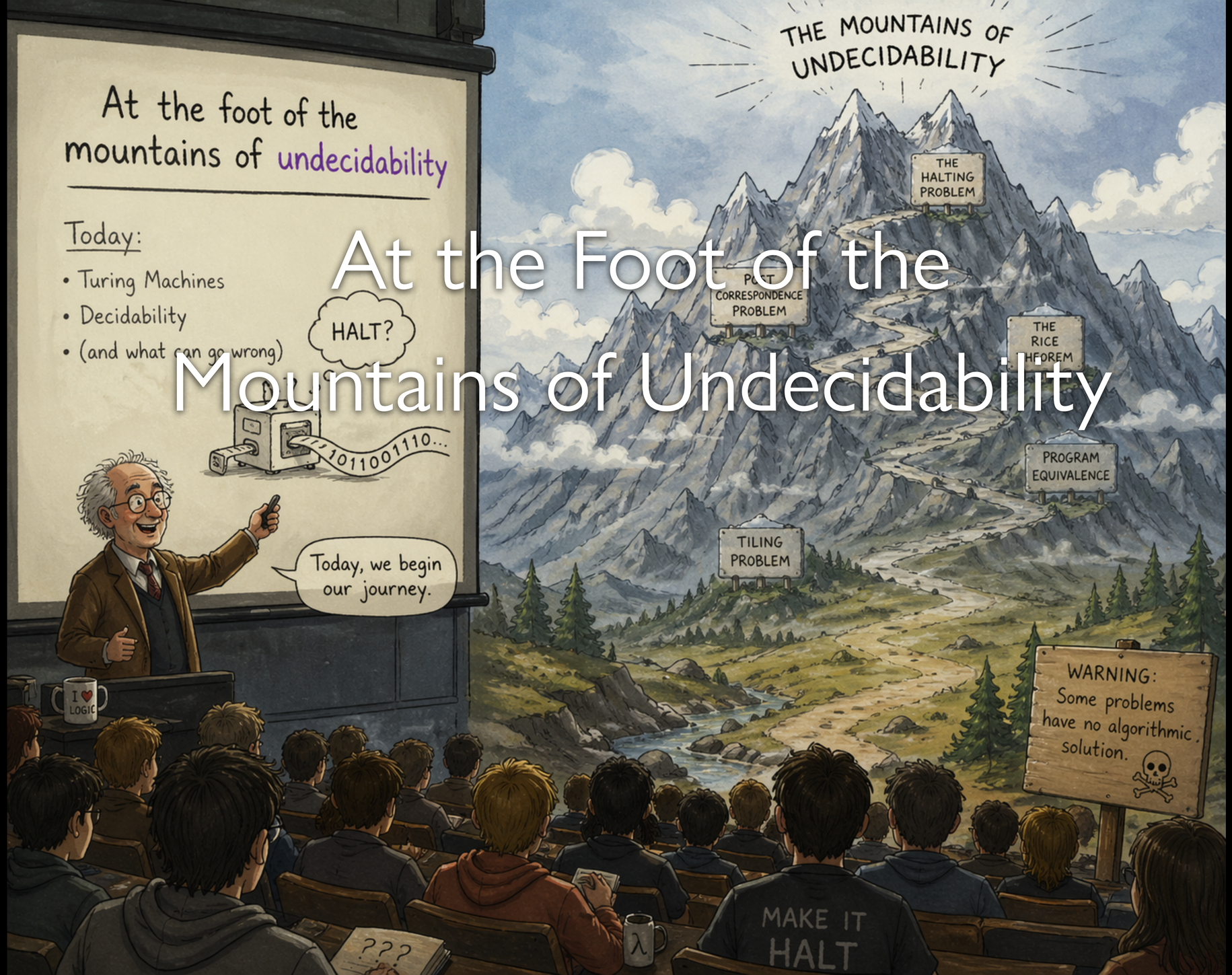
CMPU 240 · Theory of Computation

At the Foot of the Mountains of Undecidability

23 April 2026


At the Foot of the
Mountains of Undecidability



At the Foot of the Mountains of Undecidability

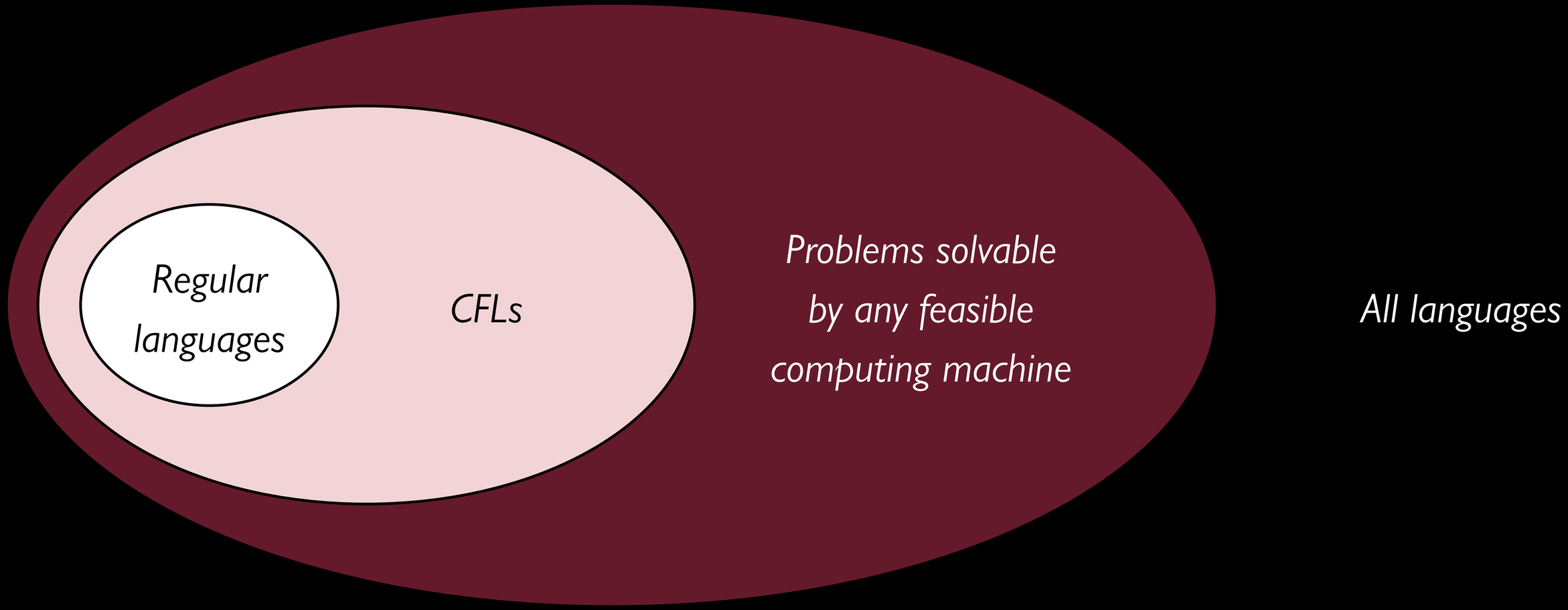
“What problems can we solve with a computer?”

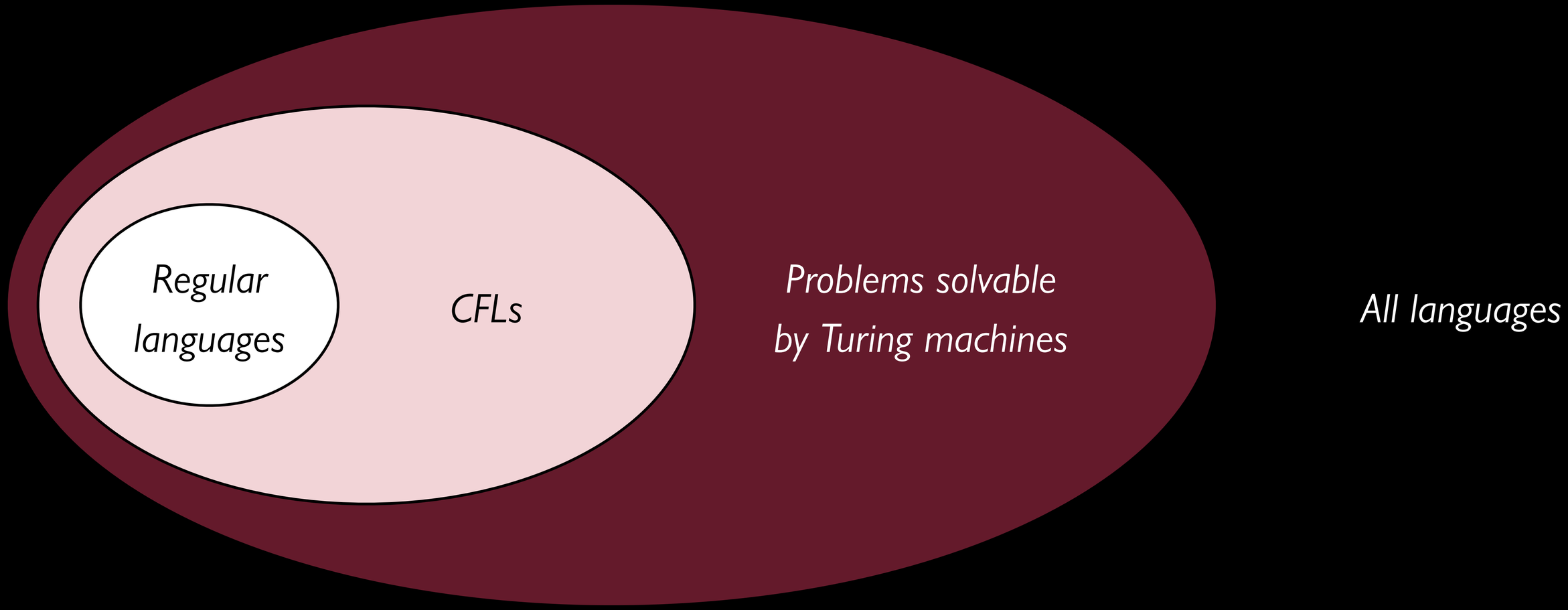
“What problems can we solve with a computer?”

What kind of computer?

The *Church–Turing Thesis* claims that:

> Every feasible method of computation is either equivalent to or weaker than a Turing machine.


Regular languages
CFLs
Problems solvable by any feasible computing machine
All languages


Regular languages
CFLs
Problems solvable by Turing machines
All languages

Because of the Church–Turing Thesis, we can start to be less detailed with our TM descriptions.

A *high-level description* of a Turing machine – as in the textbook – is a description like this:

M = "On input *x*:

Repeat the following:

If $|x| \leq 1$, accept.

If the first and last symbols of *x* aren't the same, reject.

Remove the first and last characters of *x*."

High-level descriptions are just a kind of pseudocode!

M = "On input *x*:

Repeat the following:

If $|x| \leq 1$, accept.

If the first and last symbols of *x* aren't the same, reject.

Remove the first and last characters of *x*."

```python
def M(x: str) -> bool:
    while True:
        if len(x) <= 1:
            return True
        if x[0] != x[-1]
            return False
        x = x[1:-1]
```

Every Turing machine

receives some input,

does some work, then

(optionally) accepts or rejects.

So, we can model a Turing machine as a computer program where

the program's logic is written in a normal programming language, and

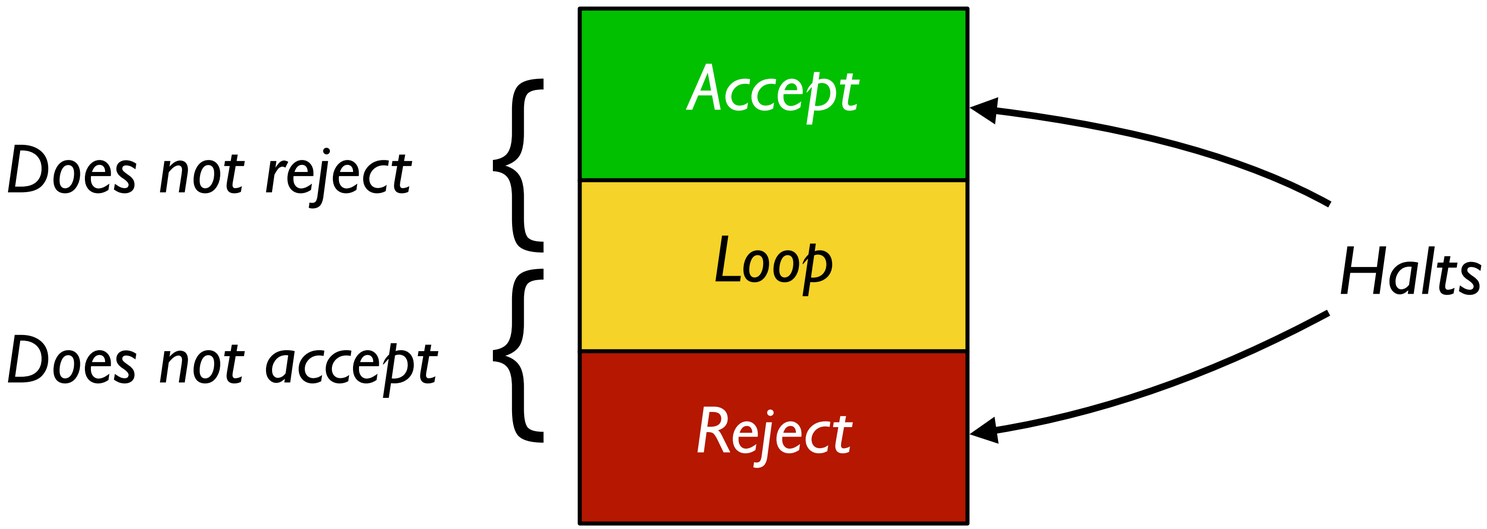
the program (optionally) returns `True` to immediately accept the input or returns `False` to immediately reject the input.

“What problems can we solve with a computer?”

What does it mean to “solve” a problem?

Unlike finite automata, which automatically halt after reading the input, Turing machines keep running until they explicitly enter an accept or reject state.

As such, it's possible for a Turing machine to run forever without accepting or rejecting.


Does not reject
Does not accept
Accept
Loop
Reject
Halts

Recognizers and recognizability

A TM M is called a ***recognizer*** for a language L over Σ if the following statement is true:

$\forall\, w \in \Sigma^* \,.\, (w \in L \Leftrightarrow M \text{ accepts } w)$

This is a weak notion of "solving a problem":

If you are absolutely certain that $w \in L$, then running a recognizer for L on w will confirm this: Eventually M will accept w!

If you don't know whether $w \in L$, running M on w may never tell you anything: M might loop on w, but you can't differentiate between "it'll never give an answer" and "just wait a bit longer"!

Deciders and decidability

A TM M is called a ***decider*** for a language L over Σ if the following statements are true:

$\forall\, w \in \Sigma^* \,.\, (w \in L \Leftrightarrow M \text{ accepts } w)$

M is a recognizer for L

$\forall\, w \in \Sigma^* \,.\, M \text{ halts on } w.$

M halts on all inputs

This is a strong notion of "solving a problem":

> If you don't know whether $w \in L$, running M on w will (eventually) give you an answer to that question.

R and **RE** languages

The class **R** consists of all decidable languages.

The class **RE** consists of all recognizable languages.

A feel for **R** and **RE**

You want to see if the hailstone sequence terminates for some $n \in \mathbb{N}$.

*An **RE** perspective*: Run the hailstone sequence starting at n. If it stops, return true. But if the hailstone sequence doesn't terminate, you'll never learn this.

*An **R** perspective*: Perform some calculation on the number n that determines whether the hailstone sequence terminates, but without actually running the hailstone sequence.

A feel for **R** and **RE**

You have a DFA. You want to see if the DFA accepts any strings of the form $a^n b^n$.

A feel for **R** and **RE**

You have a DFA. You want to see if the DFA accepts any strings of the form $a^n b^n$.

*Not whether the **language** of the DFA is $a^n b^n$, which we proved is impossible, just whether it accepts **any** string of this form!*

A feel for R and RE

You have a DFA. You want to see if the DFA accepts any strings of the form $a^n b^n$.

An RE perspective: Run the DFA on a^0b^0, a^1b^1, a^2b^2, etc. If the DFA ever accepts, return true. But, if not, you may never learn this.

An R perspective: Look at the structure of the DFA and, somehow, determine whether it accepts any strings of this form, but without running the DFA on all of them.

A feel for R and RE

Say you're working on a CS assignment. You wonder if there's any input that will make your program crash.

An RE perspective: Try running the program on every possible input. If you see it crash, return true. If it never crashes, you will never learn this.

An R perspective: Look at the source code and somehow determine, with 100% certainty, whether the program will ever crash.

A feel for **R** and **RE**

You have an *X*. You want to see if there's a *Y* where *X* and *Y* go well together.

An RE perspective: List all the *Y*s in some order and check if *X* and *Y* go well together. If so, return true. If not, you might not learn anything.

An R perspective: Look at *X* and, somehow, determine whether such a *Y* exists without checking all *Y*s.

Intuition 1: Problems in **RE** are ones that can be approached by doing some sort of exhaustive search over a potentially infinite list of options.

Intuition 2: Problems in **R** are ones that can be solved *without* having to exhaustively go through an infinite set of possibilities.

R and **RE** languages

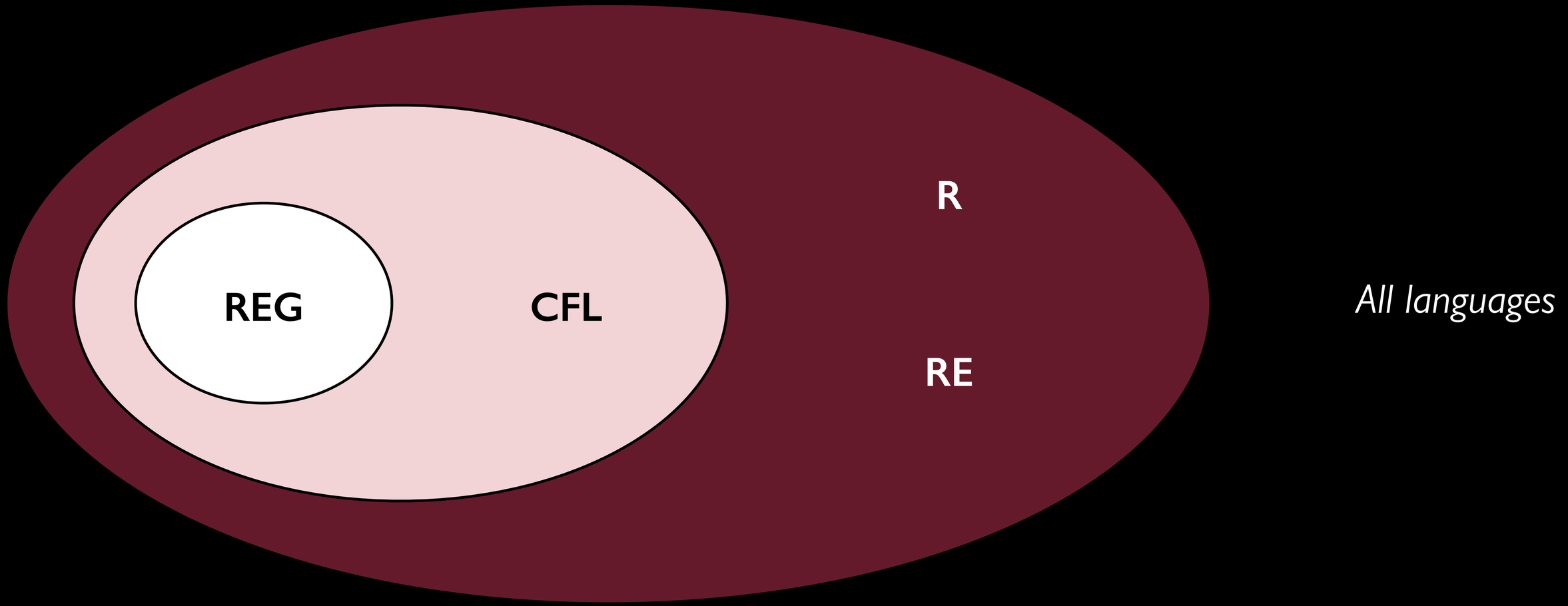
The class **R** consists of all decidable languages.

The class **RE** consists of all recognizable languages.

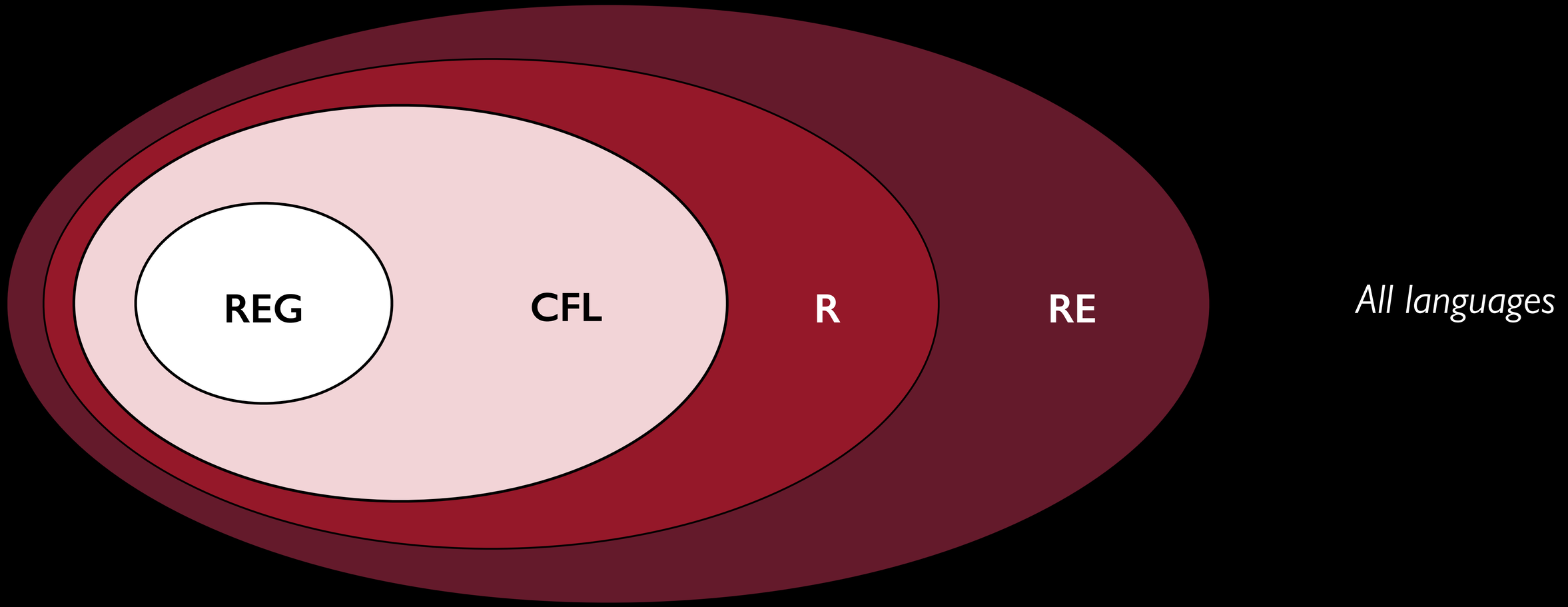
By definition, we know **R** ⊆ **RE**.

Key question: Does **R** = **RE**?

Is this right?



Or this?



What is a "problem"?

"What problems can we solve with a computer?"

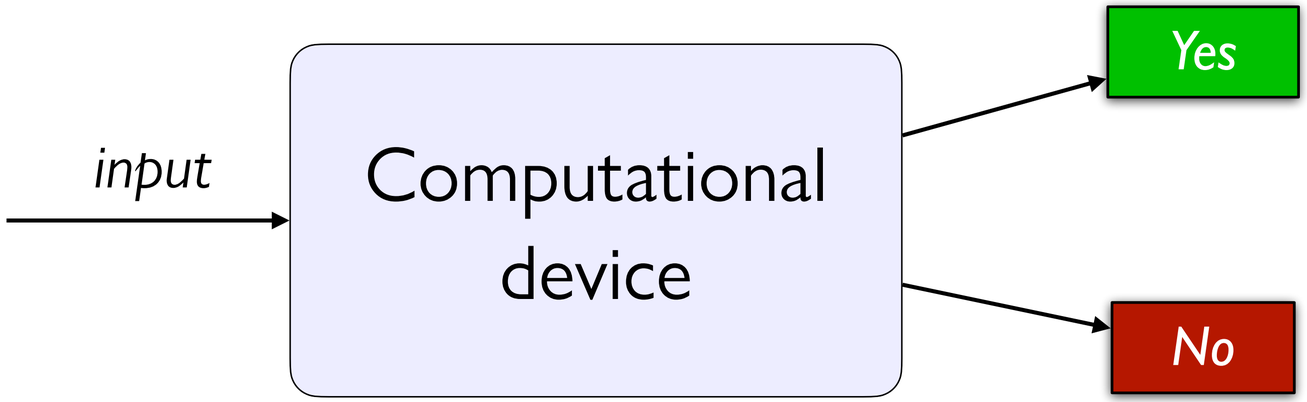
A *decision problem* is a type of problem where the goal is to answer *yes* or *no*.

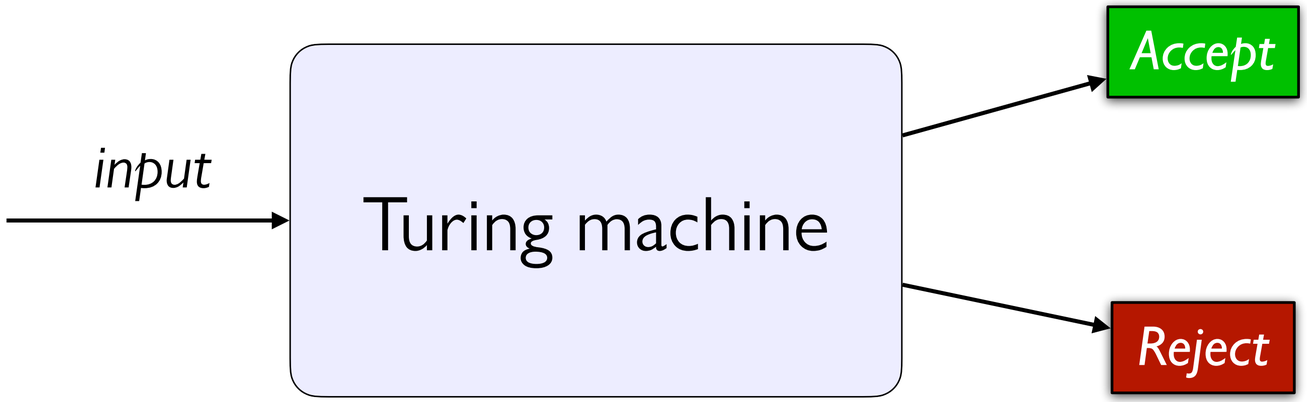
Example: *Bin Packing*

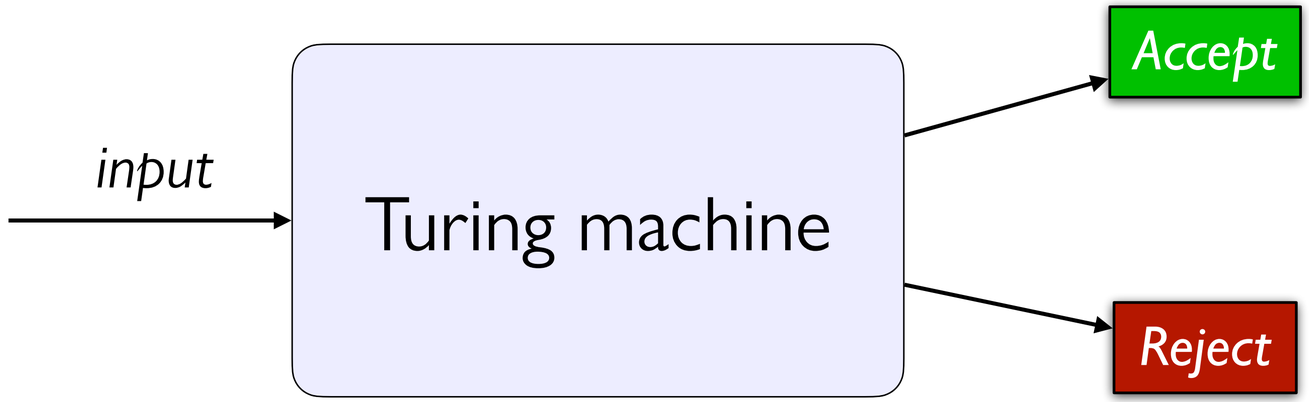
You're given a list of patients who need to be seen and how much time each needs to be seen for. You're given a list of doctors and how much free time they have. Is there a way to schedule the patients so that they can all be seen?

Example: *Route Planning*

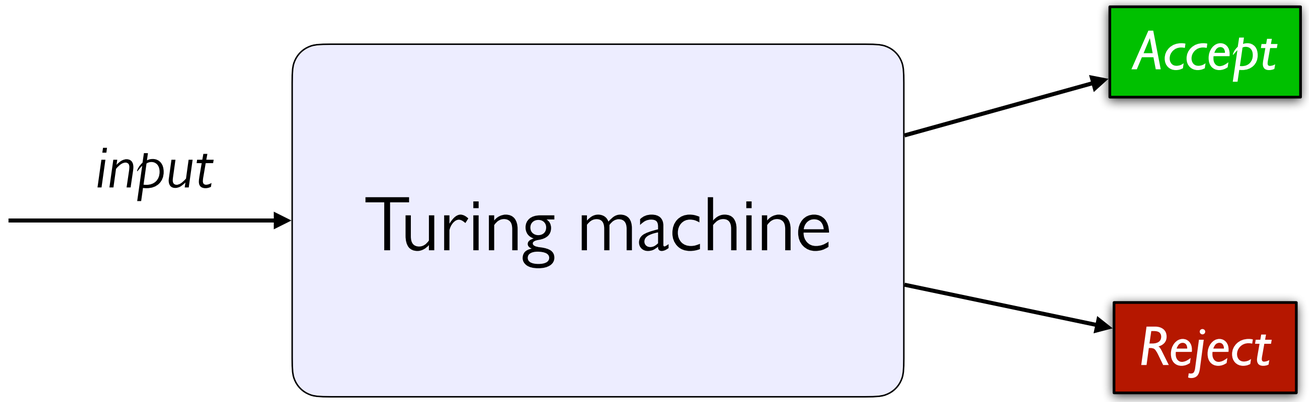
You're given a transportation grid of a city, a start location, a destination location, and information about the traffic over the course of the day. Given a time limit T, is there a way to drive from the start location to the end location in at most T hours?


input
Computational device
Yes
No

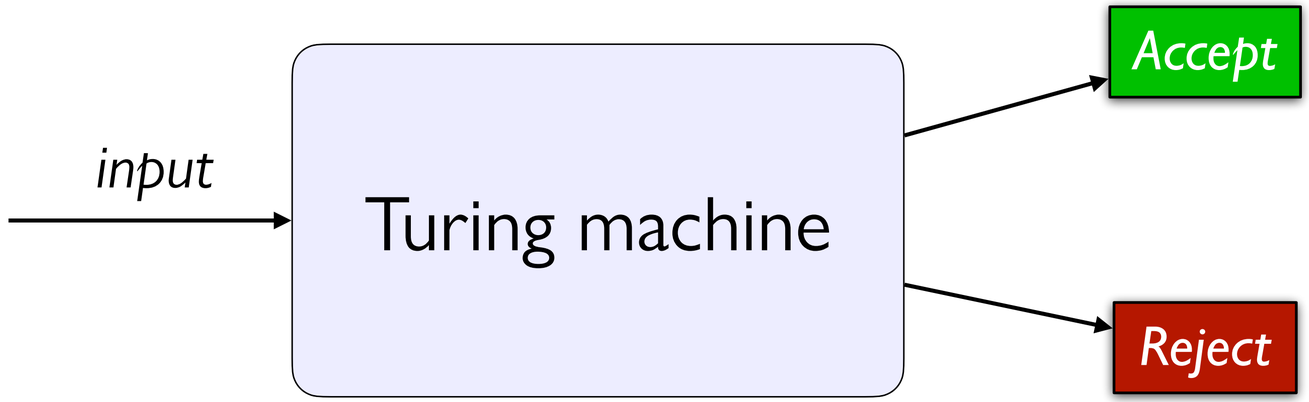

input
Turing machine
Accept
Reject



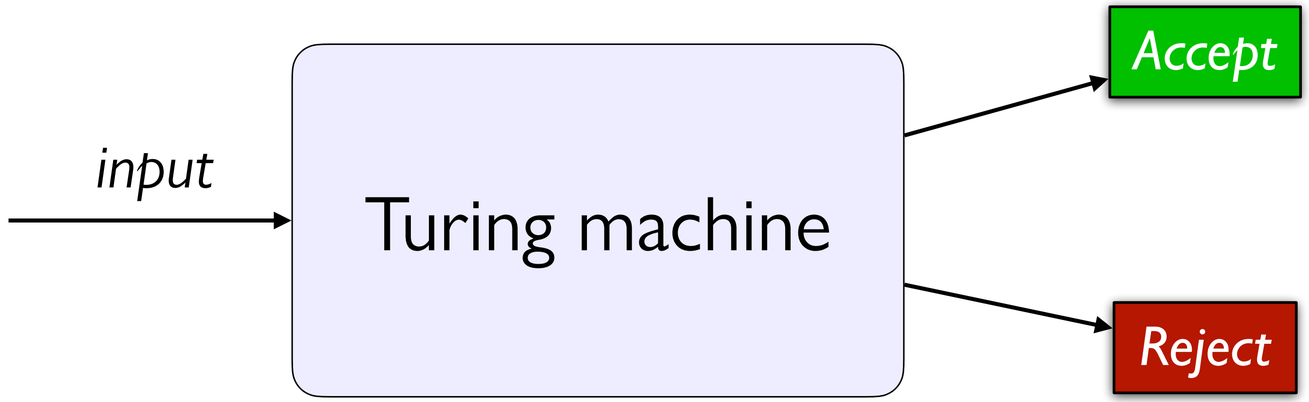

```
def some_function_name(input: str) -> bool:
    ...
```




```
def is_an_bn(input: str) -> bool:
    ...
```

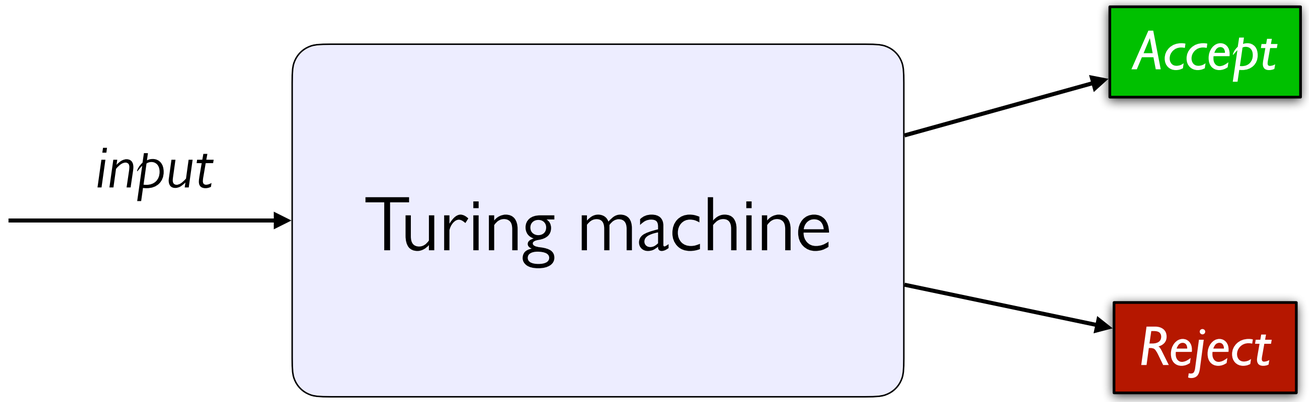



```
def is_palindrome(input: str) -> bool:
    ...
```




```
def is_fully_connected(g: Graph) -> bool:
    ...
```

How does this match our model?




```
def contains_cat(i: Image) -> bool:
    ...
```

How does this match our model?

Everything on your computer is a string over {`0`, `1`}.

Strings and objects

Think about how my computer encodes the image on the right.

Internally, it's just a series of zeros and ones on my hard drive.

Ziggy, an exemplary cat

Strings and objects

A different sequence of zeros and ones gives rise to the image on the right.

Every image can be encoded as a sequence of zeros and ones – though not all sequences of zeros and ones correspond to images!

Rigel, a fluffy example

If *Obj* is a *discrete*, *finite* mathematical object, then we'll use the notation $\langle Obj \rangle$ to refer to some reasonable encoding of that object as a string of characters.

$\langle$  $\rangle$ = `1100110100010111101001010`...

If *Obj* is a *discrete*, *finite* mathematical object, then we'll use the notation $\langle Obj \rangle$ to refer to some reasonable encoding of that object as a string of characters.

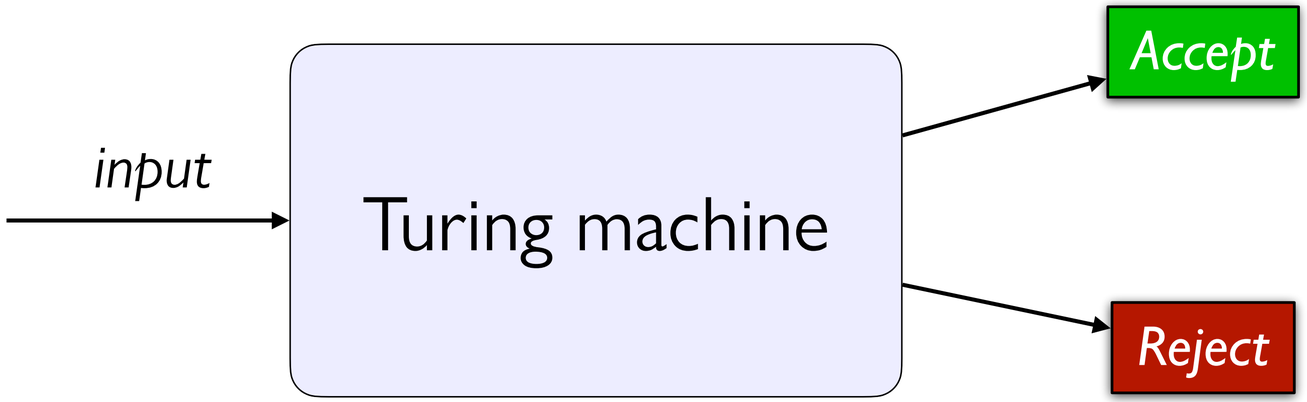
$\langle$  $\rangle$ = `000100110100111000010110`…

Object encodings

For the purposes of what we're going to be doing, we aren't going to worry about exactly *how* objects are encoded.

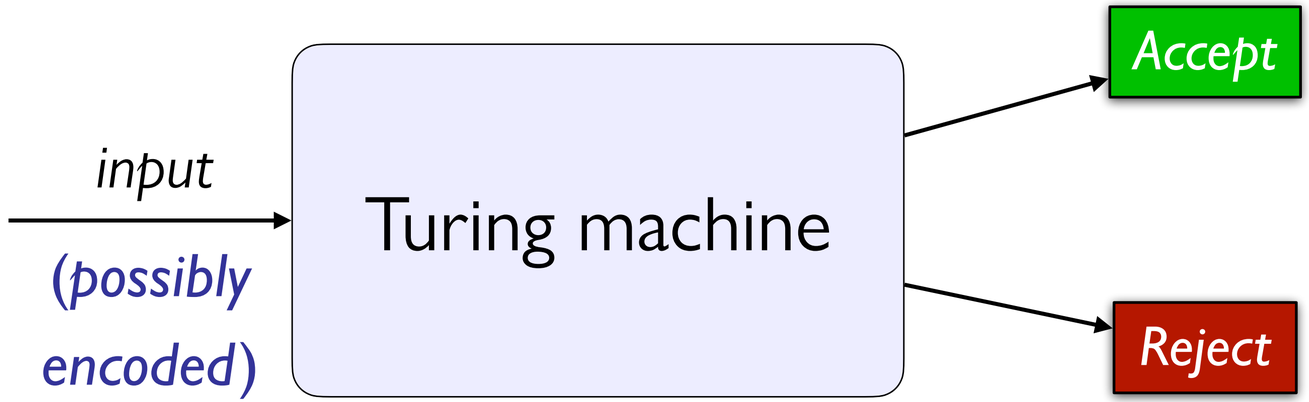
Generally we'll assume that some clever person has already figured out a way to encode what we want.

For example, we can just say, e.g., ⟨137⟩ to mean "some encoding of 137" without worrying about how it's encoded.



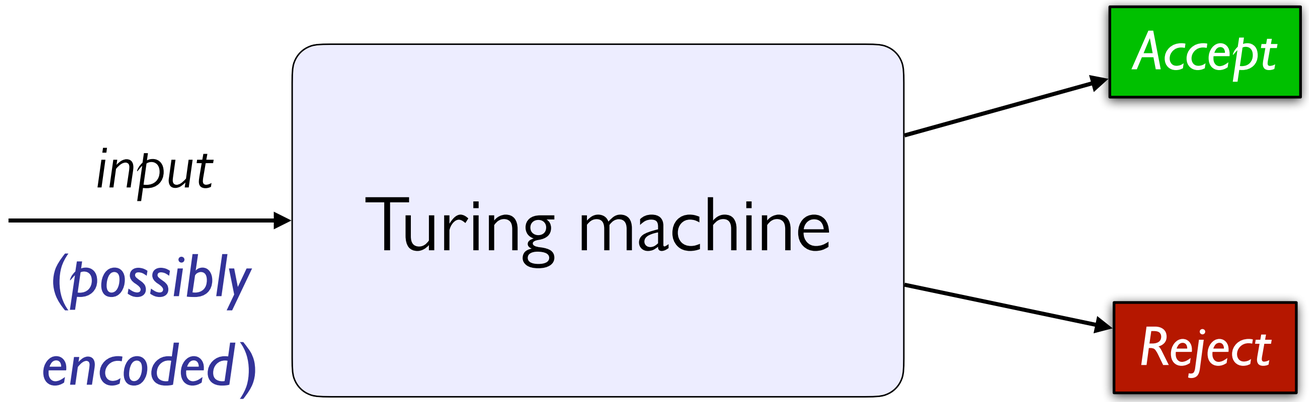

```
def contains_cat(i: Image) -> bool:
    ...
```

Internally, this is a sequence of `0`*s and* `1`*s*

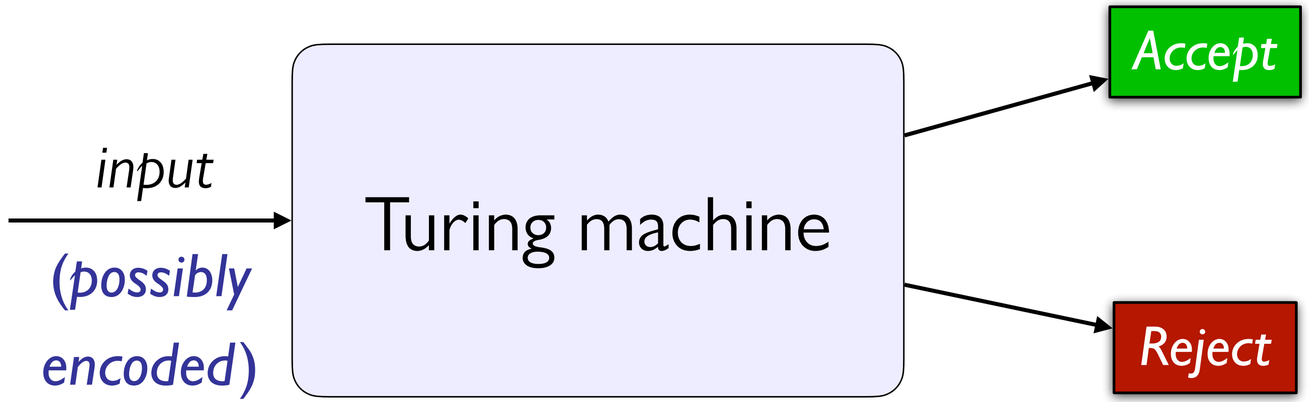



```
def contains_cat(i: Image) -> bool:
    ...
```

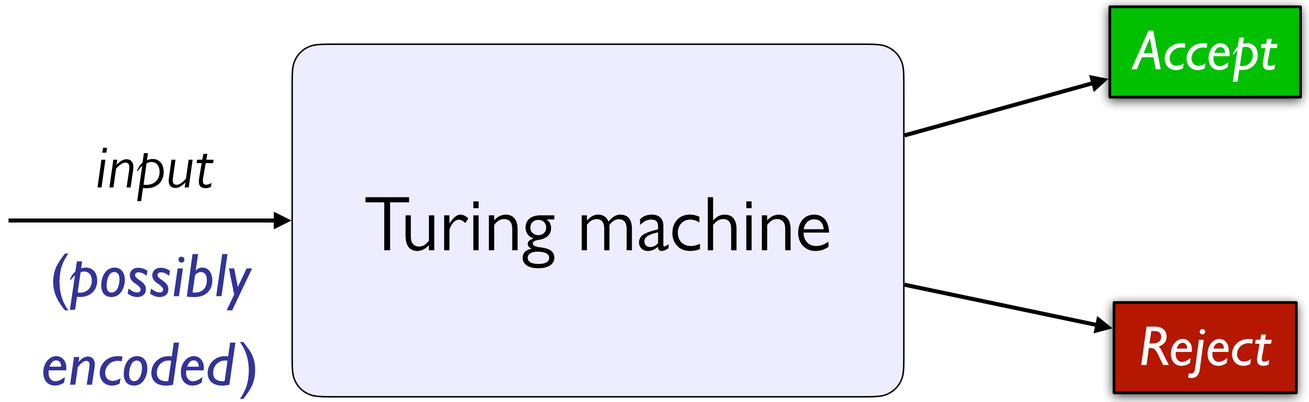
Internally, this is a sequence of `0`*s and* `1`*s*




```
def contains_cat(i: Image) -> bool:
    ...
```

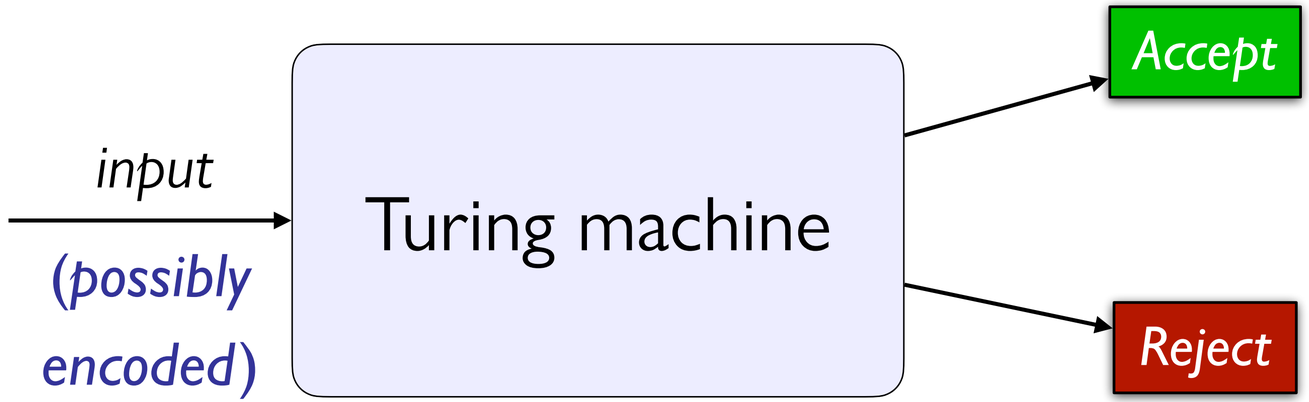



```
def is_fully_connected(g: Graph) -> bool:
    ...
```




```
def is_dominating_set(g: Graph, d: set) -> bool:
    ...
```

How does this match our model?




```
def matches_regex(s: str, r: RegEx) -> bool:
    ...
```

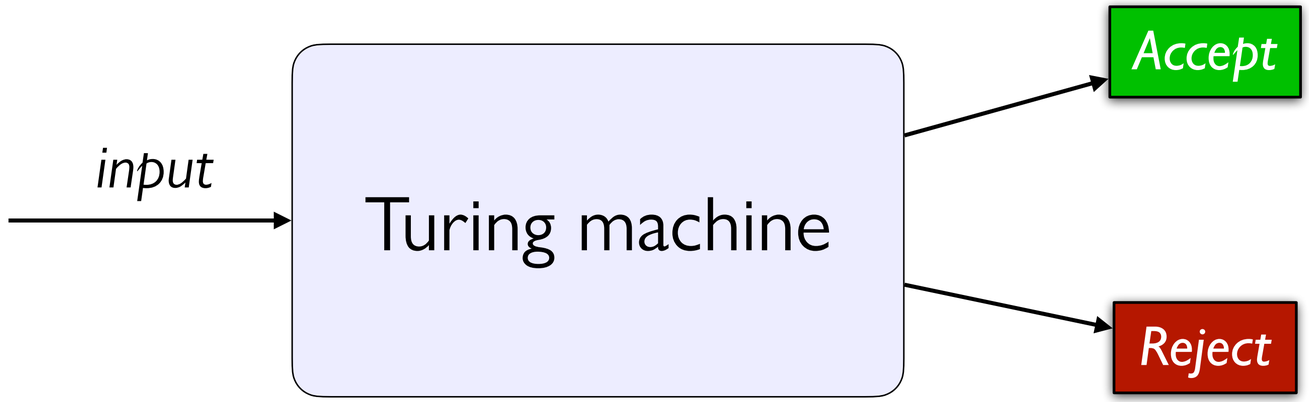
How does this match our model?

Encoding groups of objects

Given a finite group of objects, Obj_1, Obj_2, …, Obj_n, we can create a single string encoding all these objects.

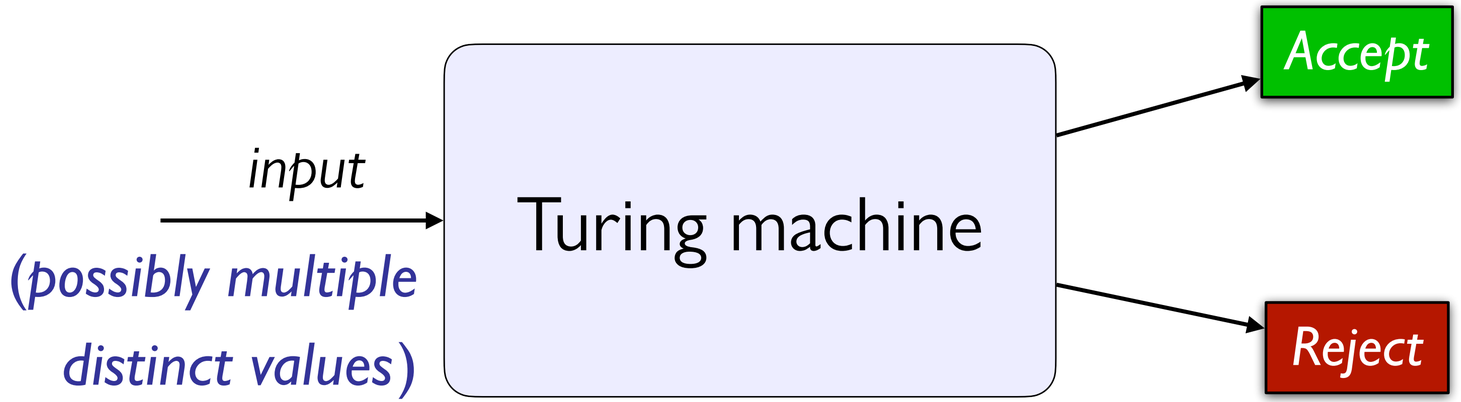
> *Intuition 1*: Think of it like a `.zip` file without the compression.
>
> *Intuition 2*: Think of it like a tuple.

We can denote the encoding of all of these objects as a single string by $\langle Obj_1, Obj_2, \ldots, Obj_n \rangle$.




```
def matches_regex(s: str, r: RegEx) -> bool:
    ...
```

These form one large bit-string.




```
def matches_regex(s: str, r: RegEx) -> bool:
    ...
```

These form one large bit-string.

Our goal is to speak of *computers solving problems*.

We model this by looking at *Turing machines recognizing languages*.

By turning any problem into an equivalent *decision problem*, this precisely captures what we're interested in.

“What problems can we solve with a computer?”

Let's think about *emergent properties*.

An emergent property of a system is a property that arises out of smaller pieces but which doesn't seem to exist in any of the individual pieces, e.g.,

- Individual neurons fire in response to particular combinations of inputs and this gives rise to human consciousness.
- Individual atoms obey the laws of quantum mechanics and just interact with other atoms, and this gives rise to literally everything.

All computing systems equal to Turing machines exhibit several surprising emergent properties.

If we believe the Church–Turing Thesis, these must be *inherent* to computation; computation can't exist without them.

These emergent properties are what ultimately make computation so interesting and powerful.

But we'll see they're also computation's Achilles heel – they're how we find concrete examples of impossible problems.

The key emergent properties of computation that we'll discuss are:

Universality:

There is a single computing device capable of performing any computation.

Self-reference:

Computing devices can ask questions about their own behavior.

We'll see that the combination of these properties leads to simple examples of impossible problems and elegant proofs of impossibility.

Universality

We've been designing Turing machines to solve specific problems.

Do you have a dedicated computer for each task you need to perform?

- Your email computer?
- Your word-processing computer?
- Your cute-cat-picture computer?

Can we make a “reprogrammable Turing machine”?

A Turing machine simulator

It's possible to program a Turing machine simulator on an unbounded memory computer.

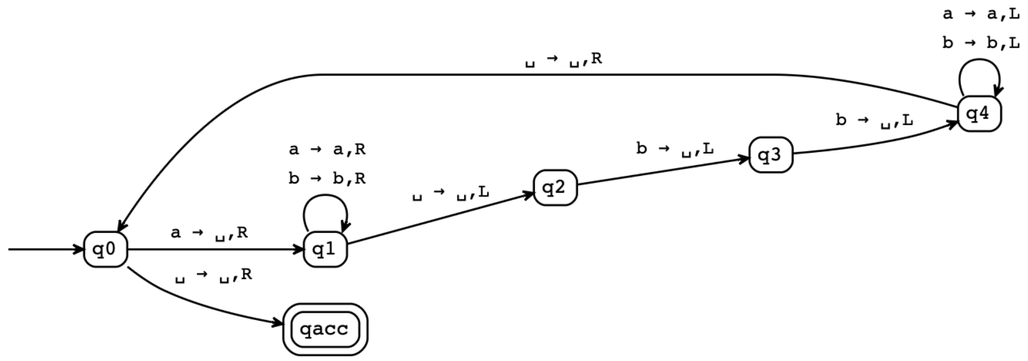
If we accept some limits on the "infinite" tape, we can even do this on a real computer.

Assignment 8 solutions.ipynb

File Edit View Insert Runtime Tools Help

Share

Commands + Code + Text Run all Connect



[ ]

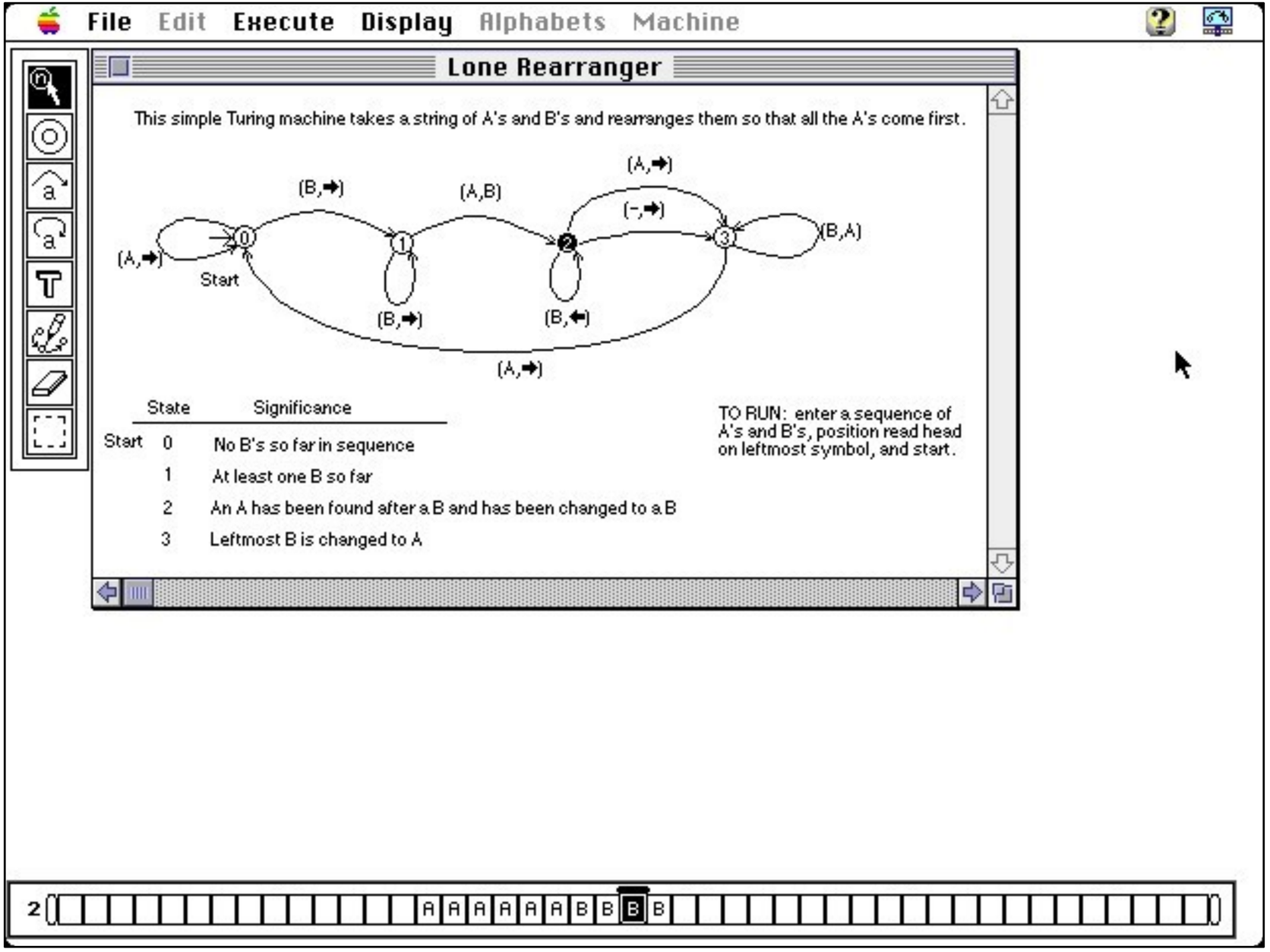
```
# This should be accepted
run(ex1, list("abb")).only_path()
```

```
q0   [a] b b
q1   ␣ [b] b
q1   ␣ b [b]
q1   ␣ b b [␣]
q2   ␣ b [b] ␣
q3   ␣ [b] ␣ ␣
q4   [␣] ␣ ␣ ␣
q0   ␣ [␣] ␣ ␣
qacc ␣ ␣ [␣] ␣
accept
```

[ ]

```
# This should be reiected
```

Variables Terminal



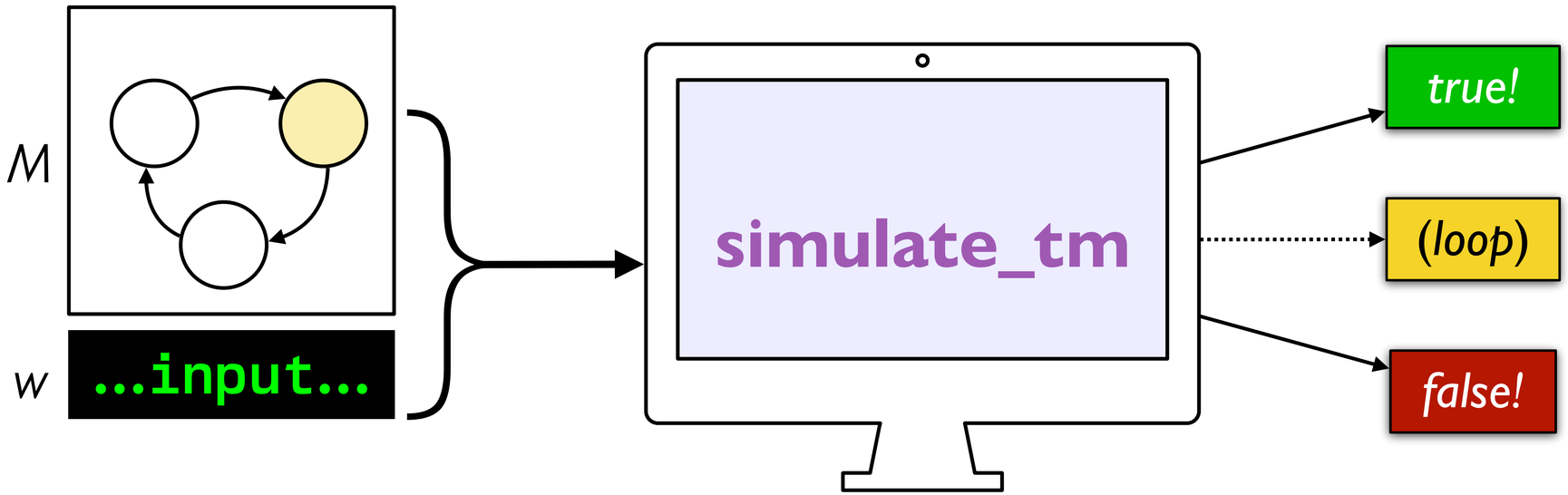
Turing's World 3.0, 1993

We can imagine a TM simulator as a procedure

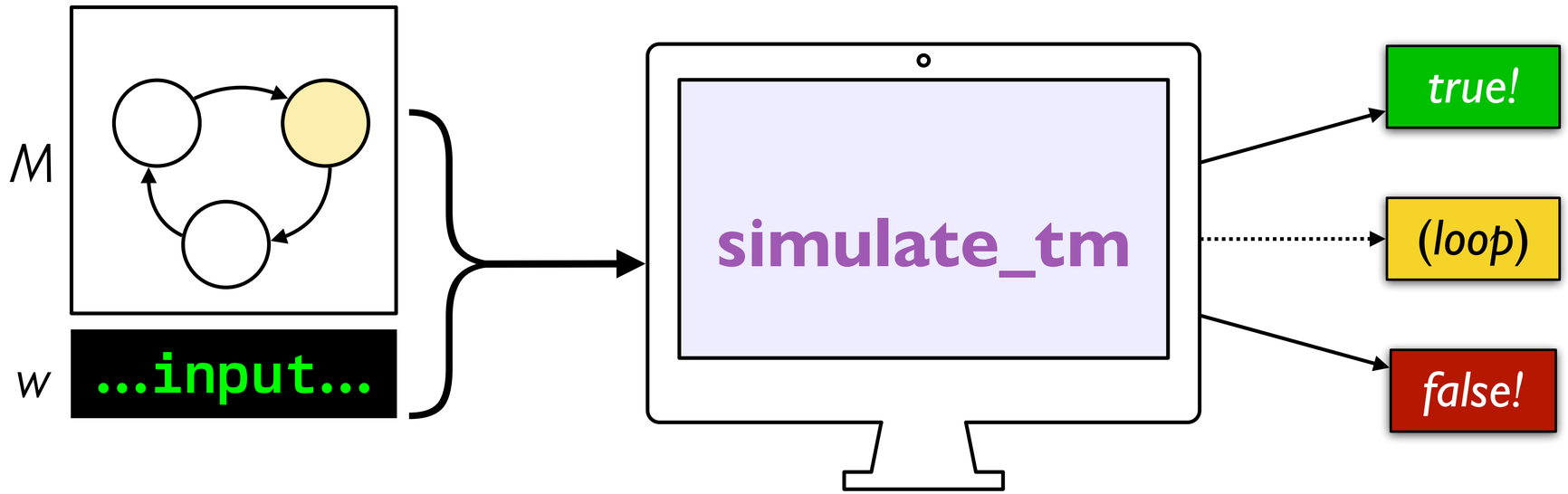
```
simulate_tm(M: TM, w: str) -> bool
```

with the following behavior:

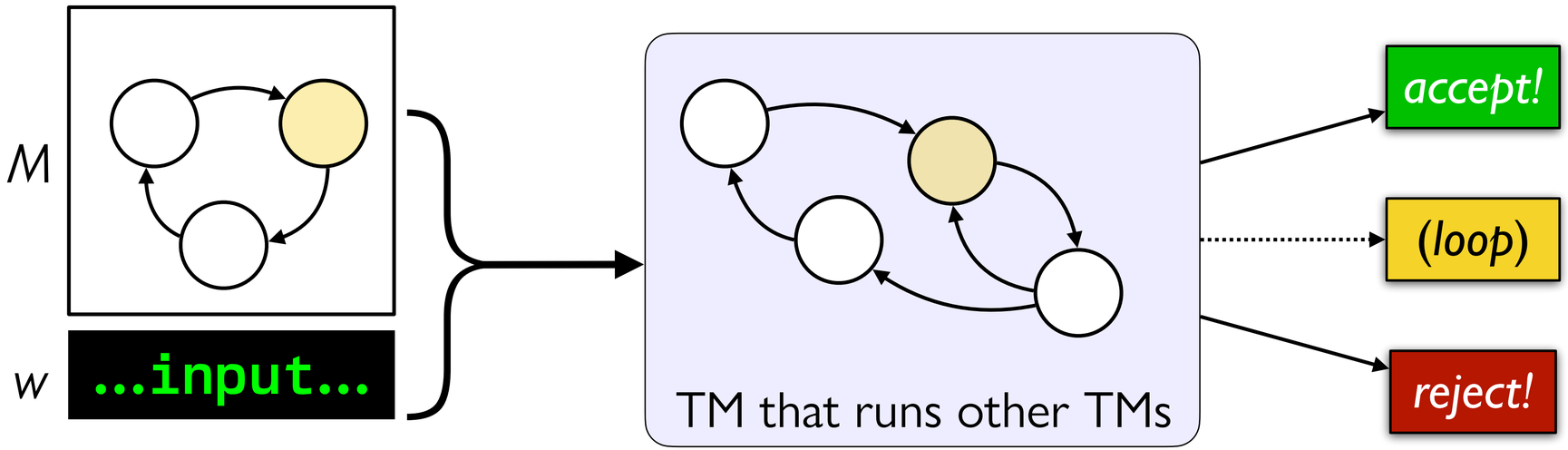
- If *M* accepts *w*, then **`simulate_tm(M, w)`** returns `True`.
- If *M* rejects *w*, then **`simulate_tm(M, w)`** returns `False`.
- If *M* loops on *w*, then **`simulate_tm(M, w)`** loops infinitely.

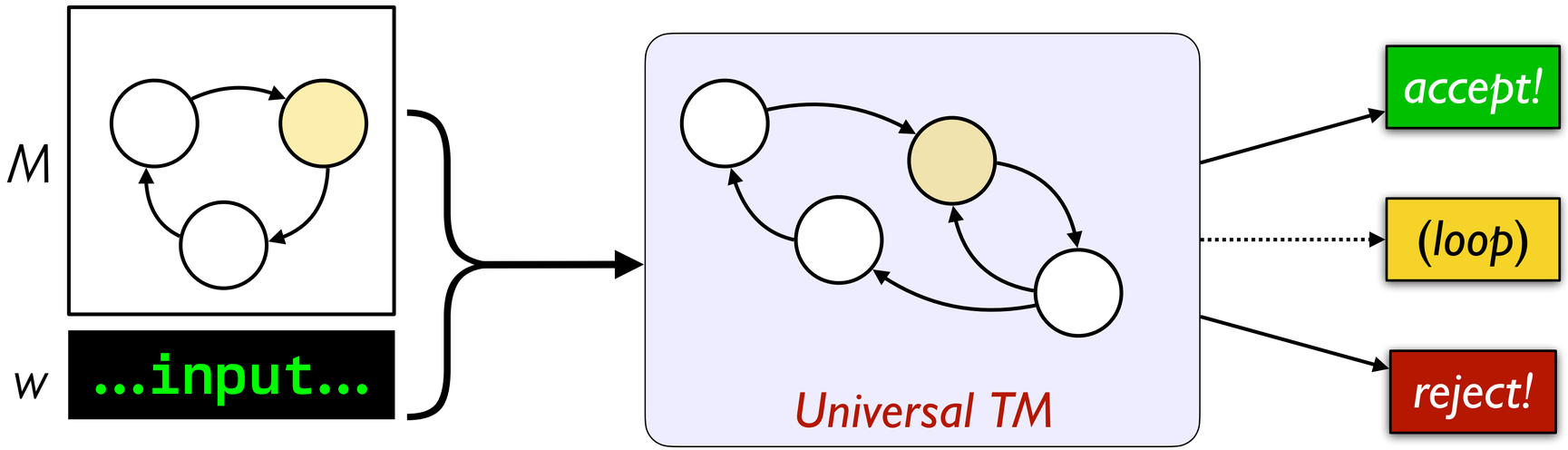

M
w
...input...
simulate_tm
true!
(loop)
false!

Anything that can be done with an unbounded-memory computer can be done with a Turing machine.



So there must be a Turing machine that has the behavior of the **`simulate_tm`** method.


M
w
...input...
TM that runs other TMs
accept!
(loop)
reject!


M
w
...input...
Universal TM
accept!
(loop)
reject!

THEOREM (Turing, 1936) There is a Turing machine *U* called the *universal Turing machine* that, when run on an input of the form $\langle M, w \rangle$, where *M* is a Turing machine and *w* is a string, simulates *M* running on *w* and does whatever *M* does on *w*.

If *M* accepts *w*, then *U* accepts $\langle M, w \rangle$.

If *M* rejects *w*, then *U* rejects $\langle M, w \rangle$.

If *M* loops on *w*, then *U* loops on $\langle M, w \rangle$.

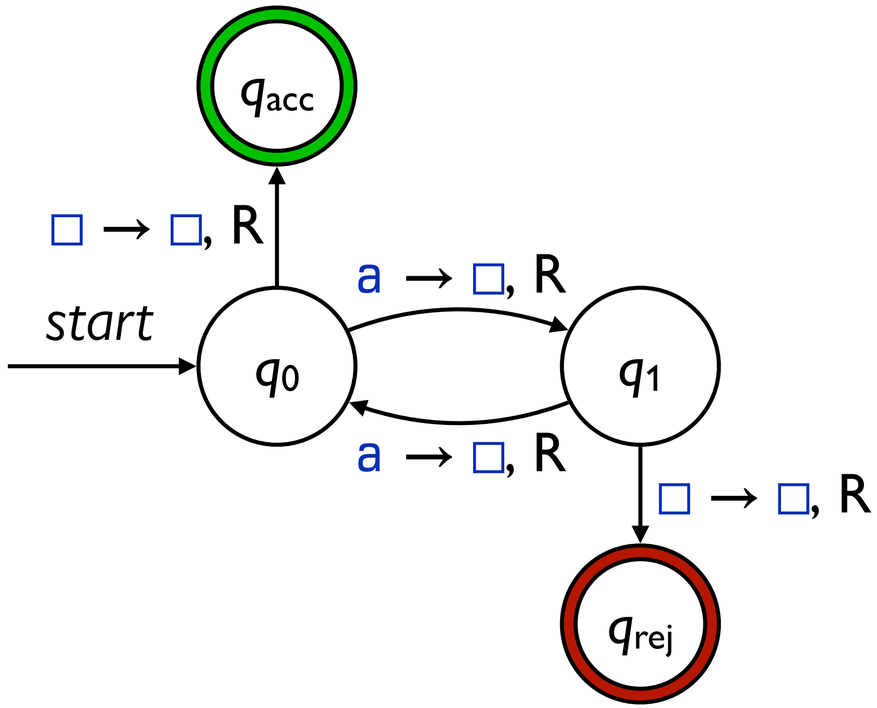
U does to input $\langle M, w \rangle$

what

M does on input w.

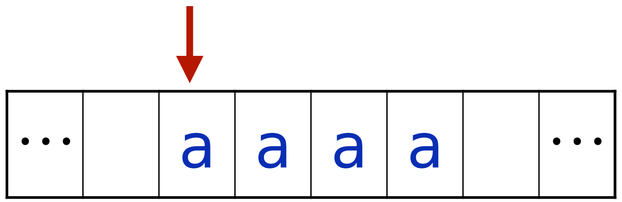
The universal Turing machine *U*, schematically

Machine *M*

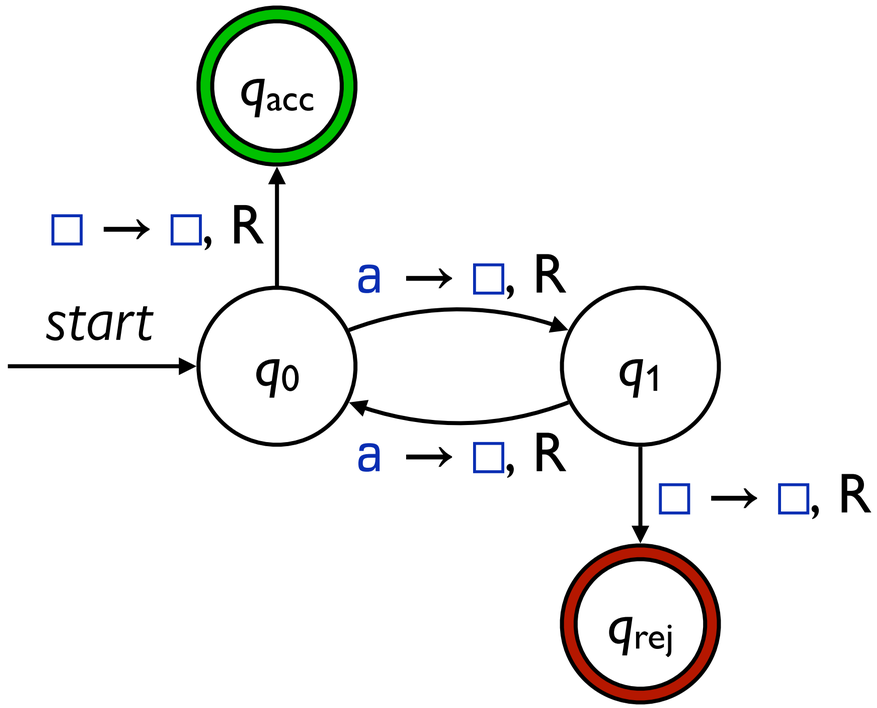


Imagine you have some machine M (like a program) that you want to run on input w.

Input *w*

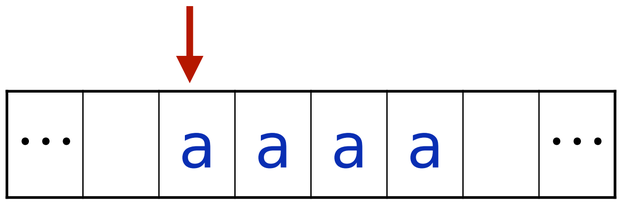


Machine *M*

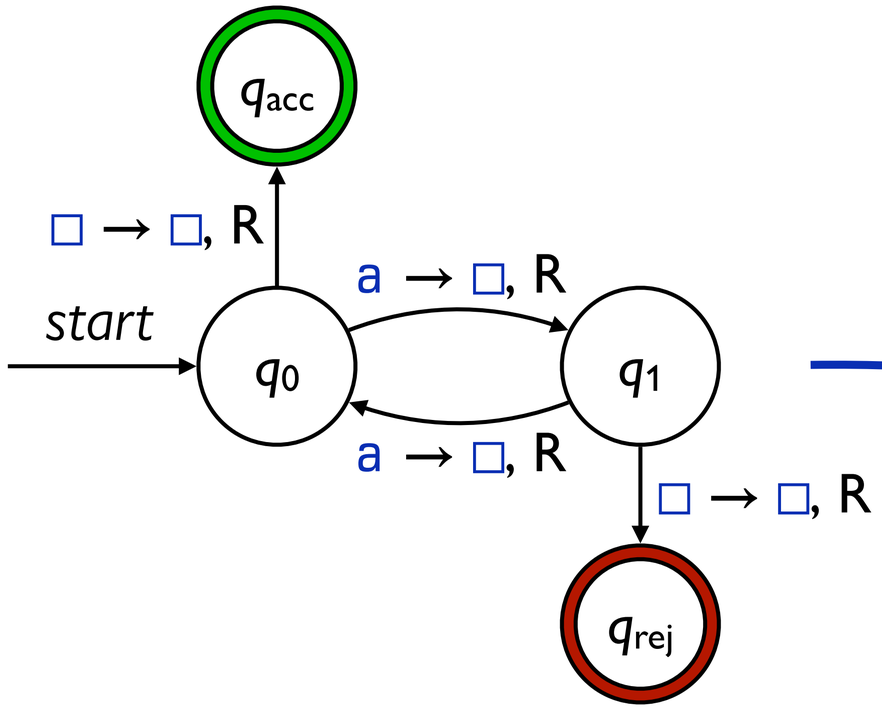


Take M and write it down as a string. (Remember how we can encode the finite-state control as table.)

Input *w*

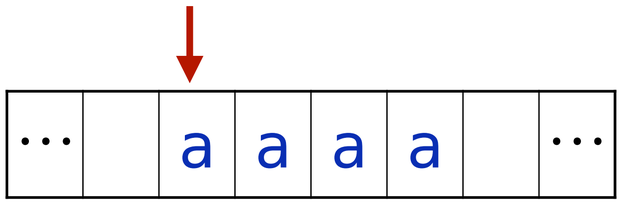


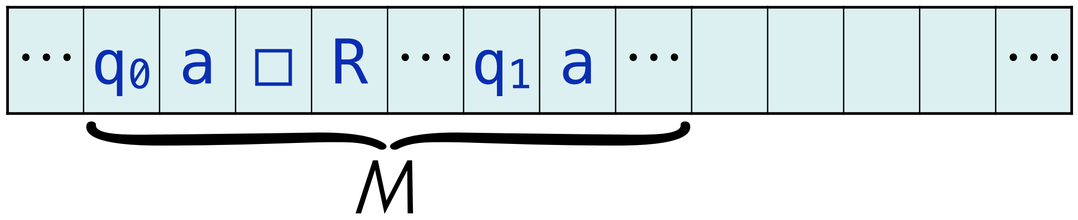
Machine *M*



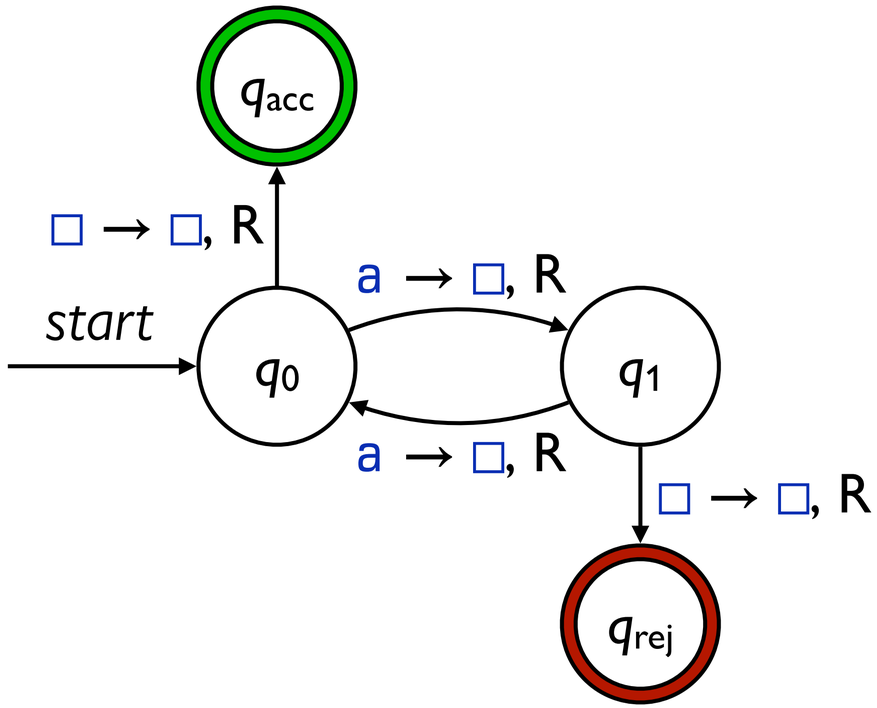
Take M and write it down as a string. (Remember how we can encode the finite-state control as table.)

Input *w*



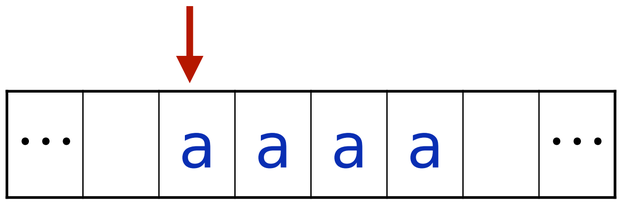


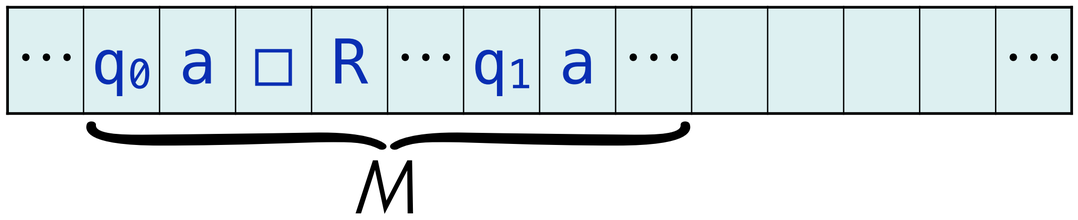
Machine *M*



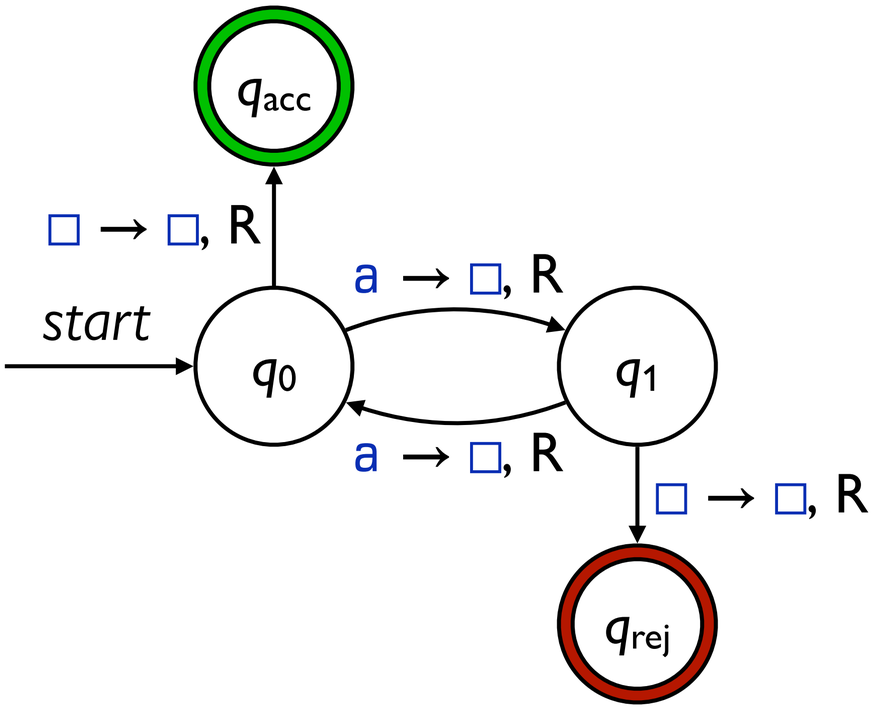
Now take your input w and write it down too.

Input *w*



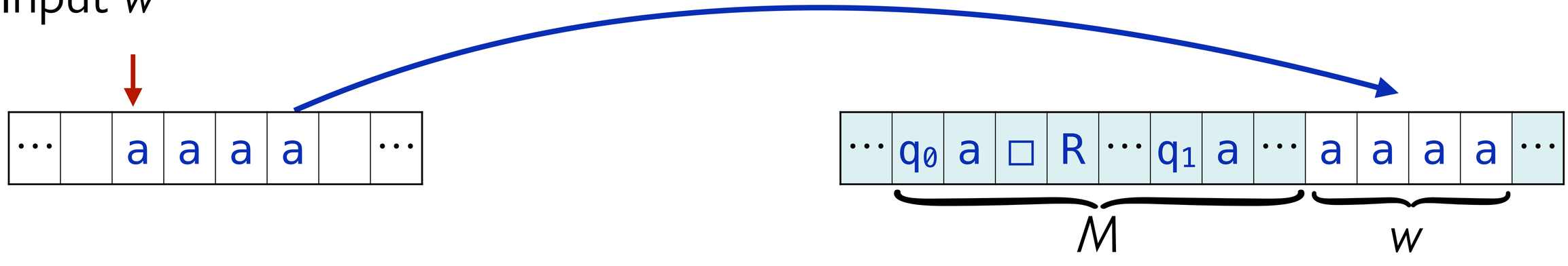


Machine *M*

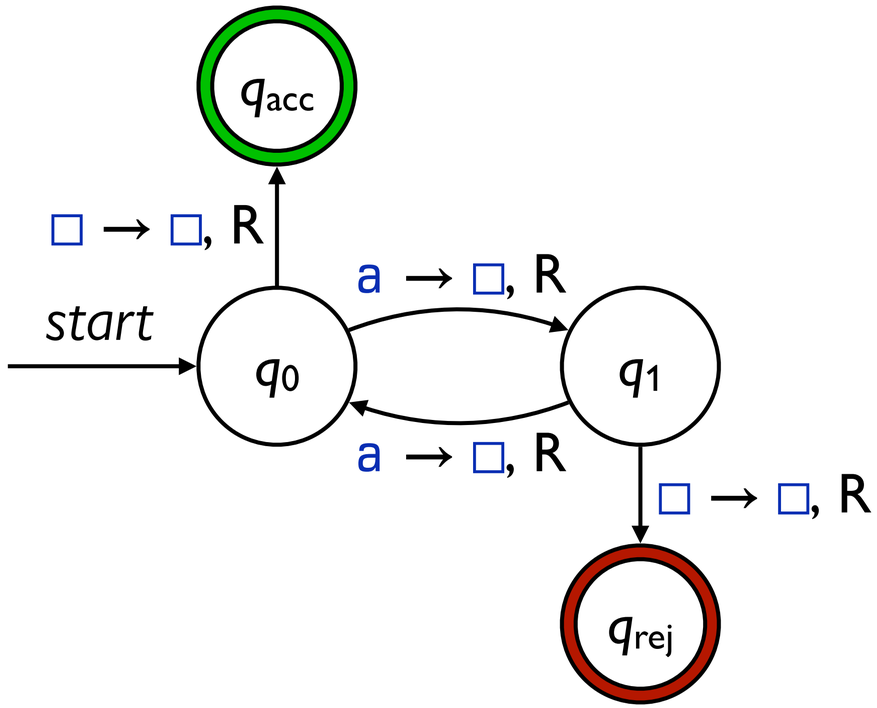


Now take your input w and write it down too.

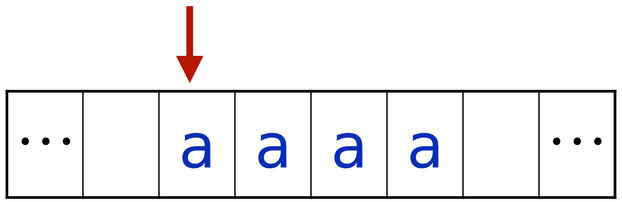
Input *w*



Machine *M*

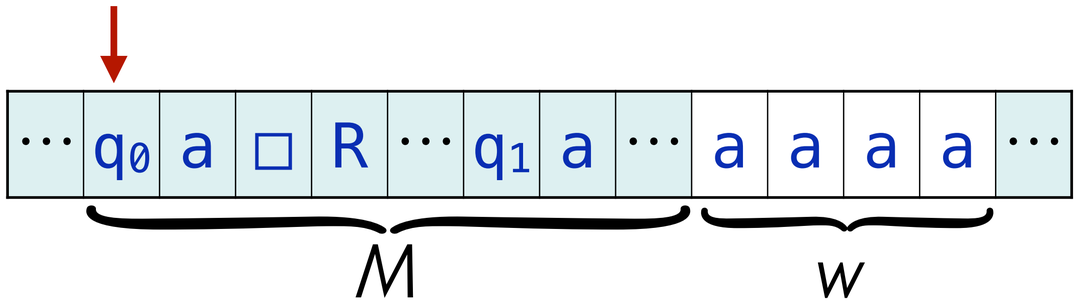


Input *w*

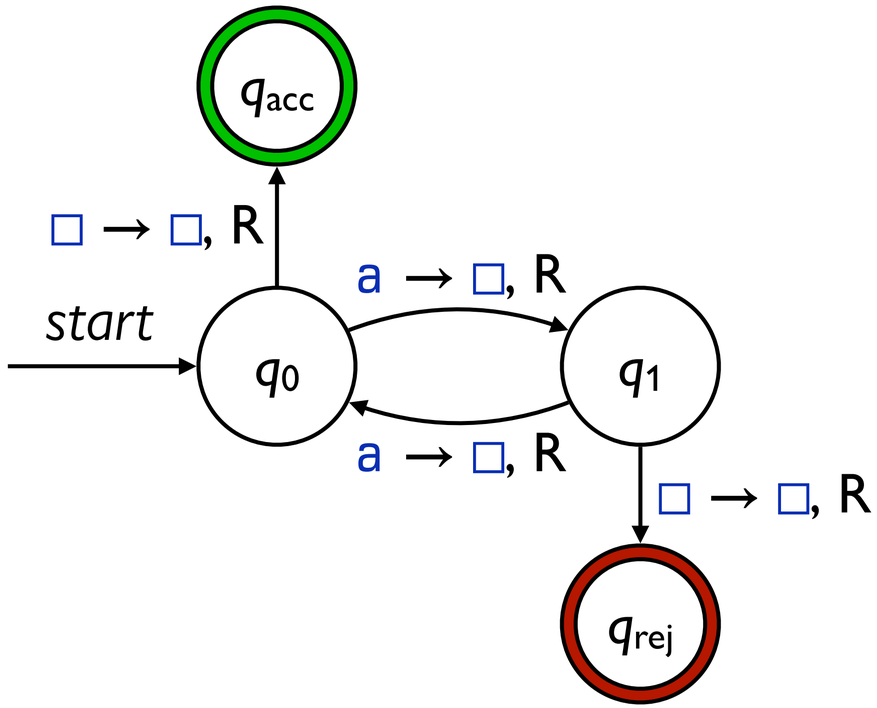


Feed this into U.

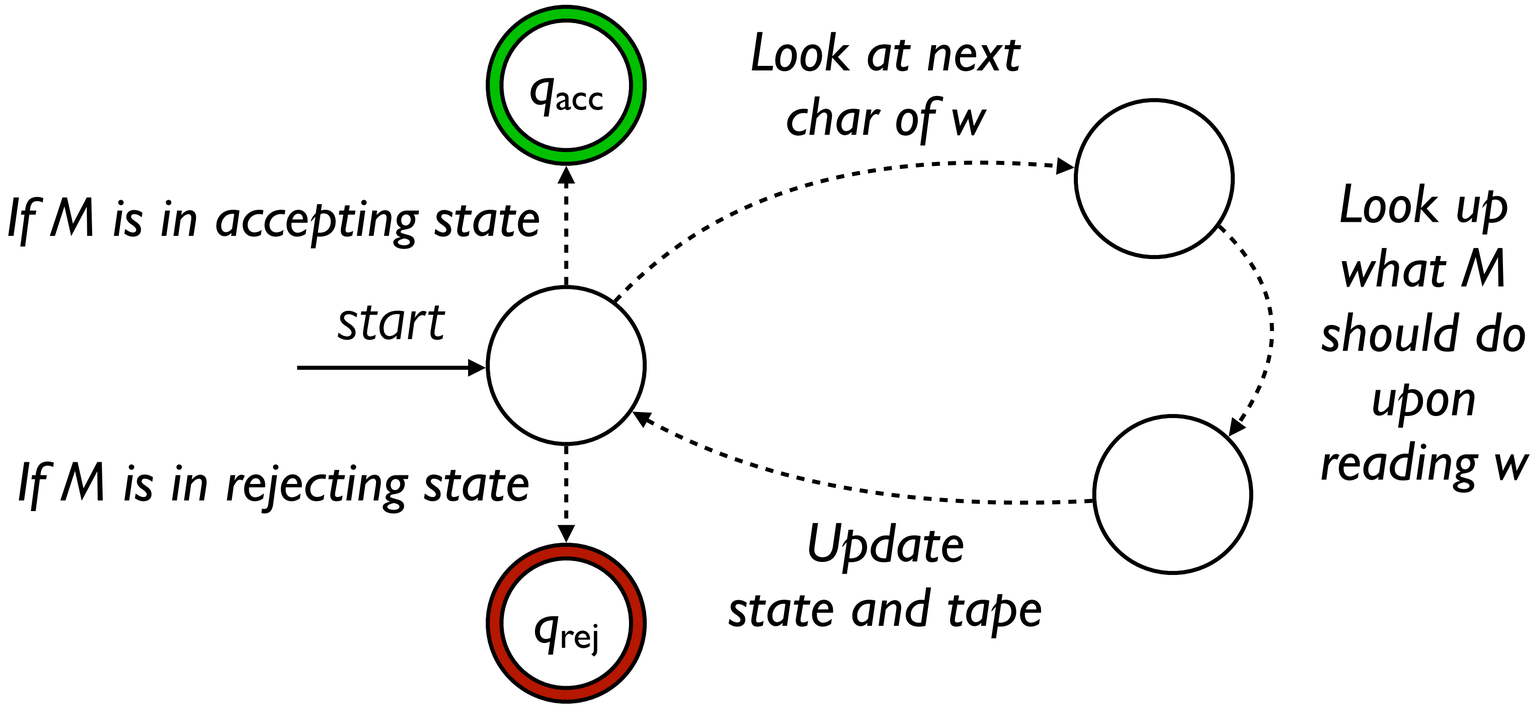
Input ⟨*M*, *w*⟩



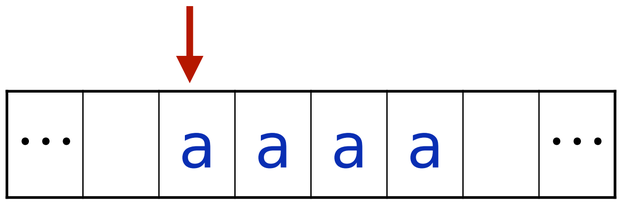
Machine *M*



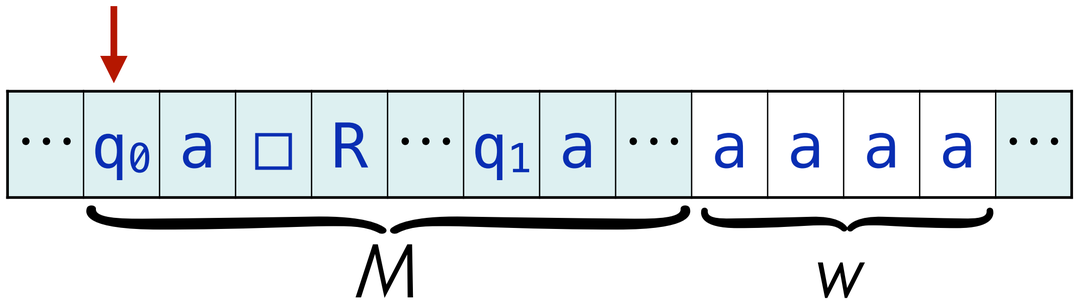
Universal TM *U*



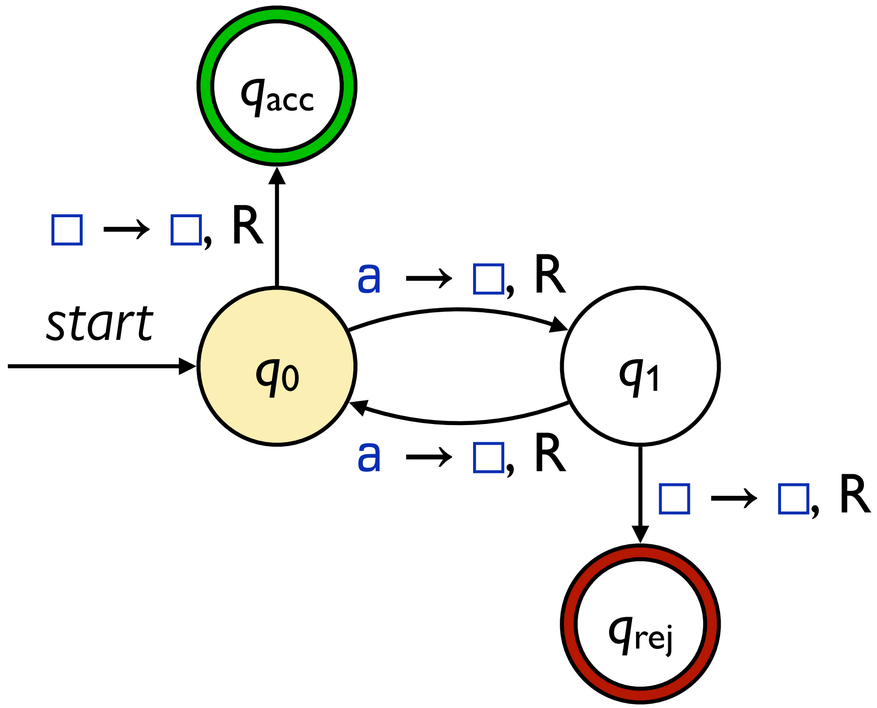
Input *w*



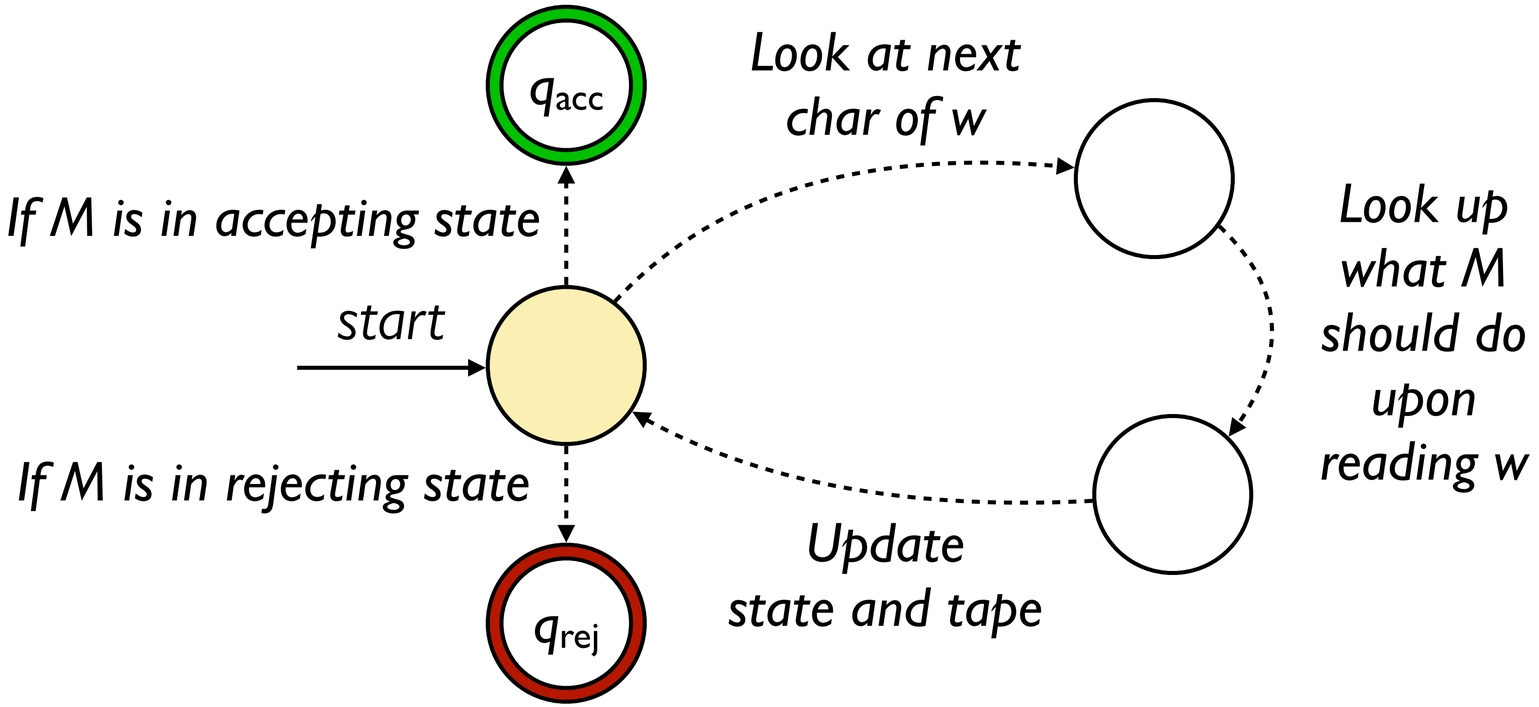
Input ⟨*M*, *w*⟩



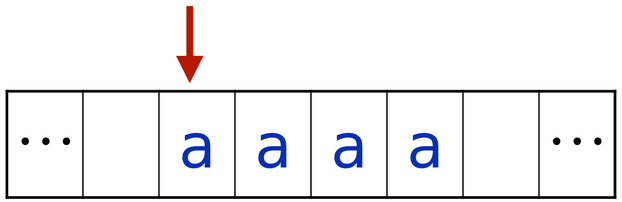
Machine *M*



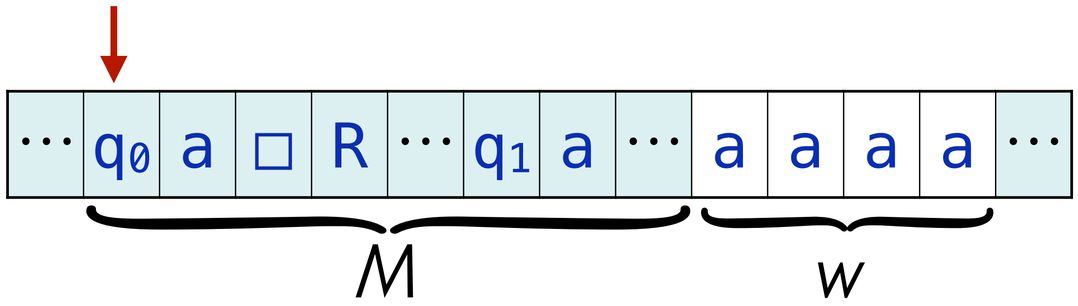
Universal TM *U*



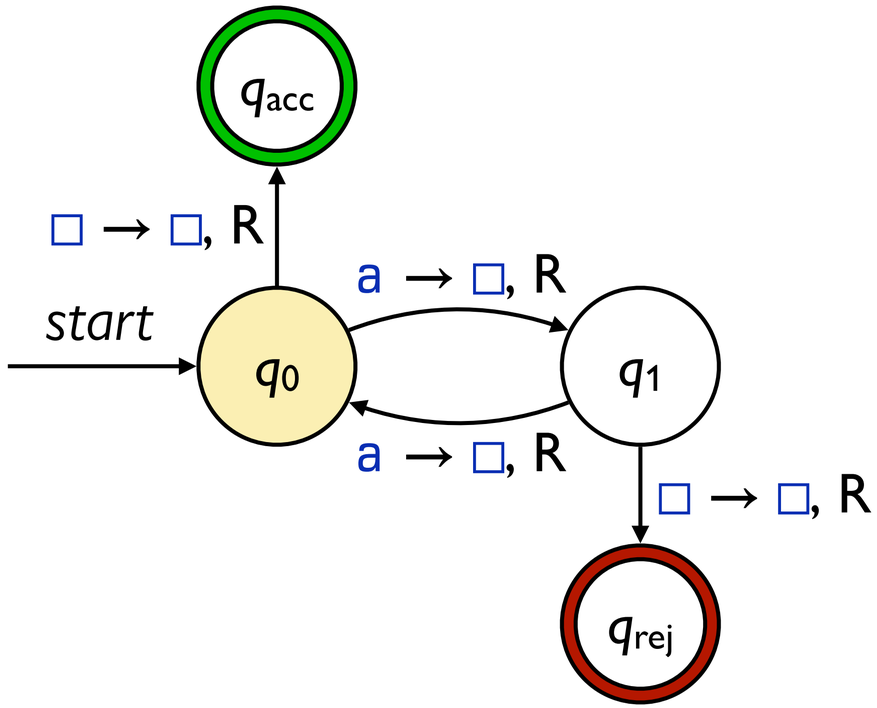
Input *w*



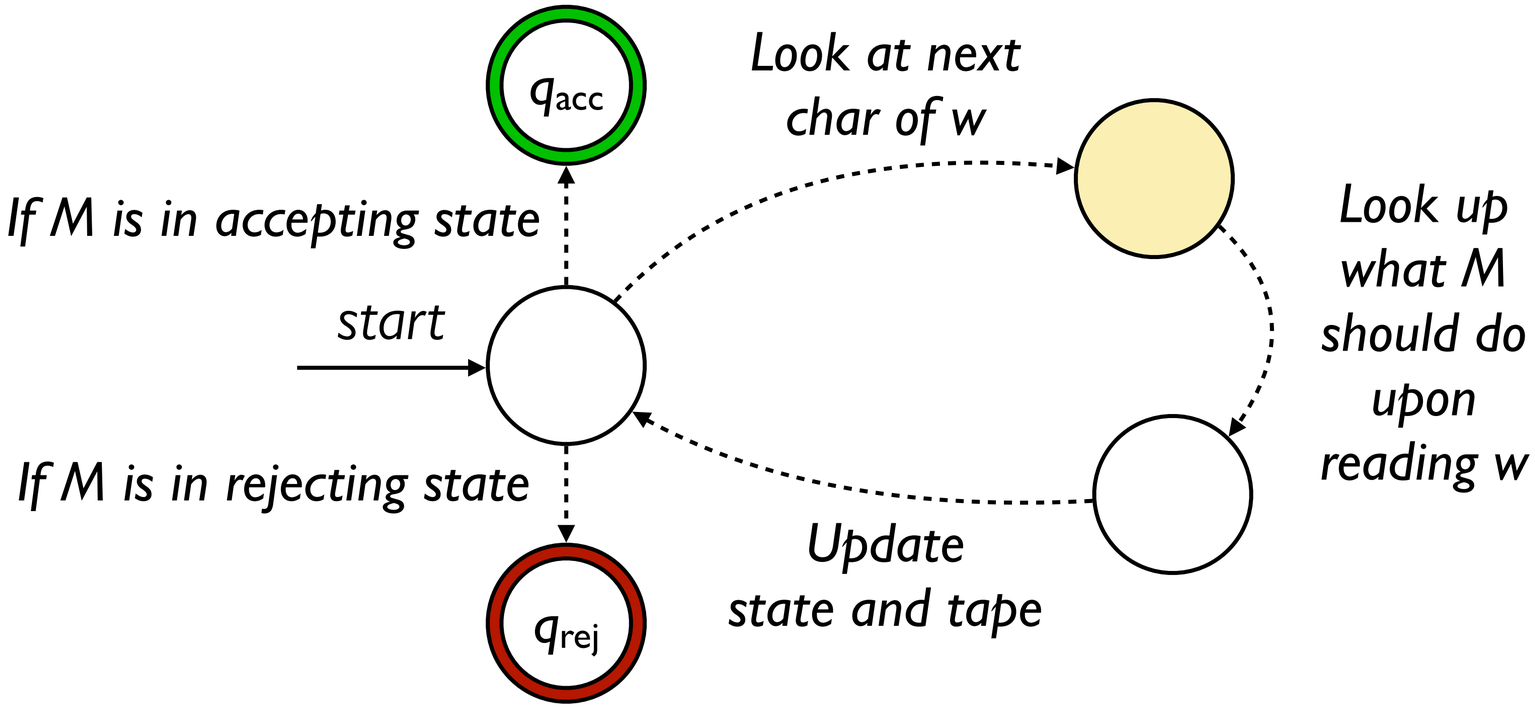
Input ⟨*M*, *w*⟩



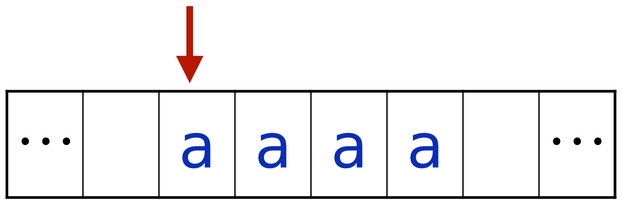
Machine *M*



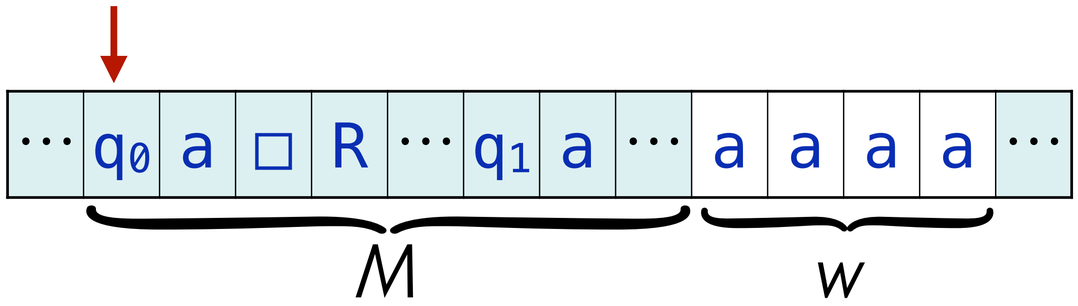
Universal TM *U*



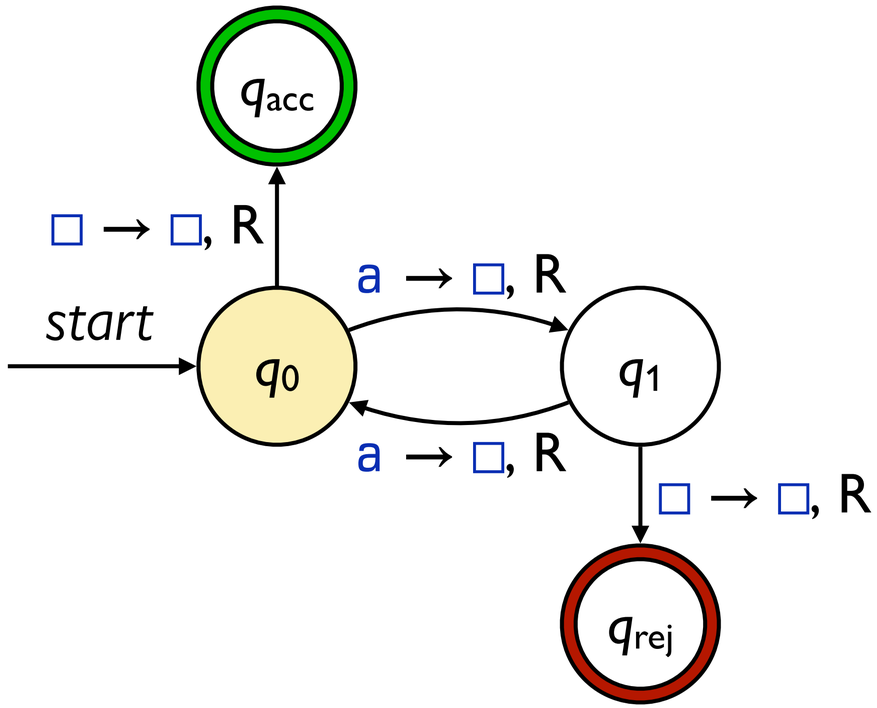
Input *w*



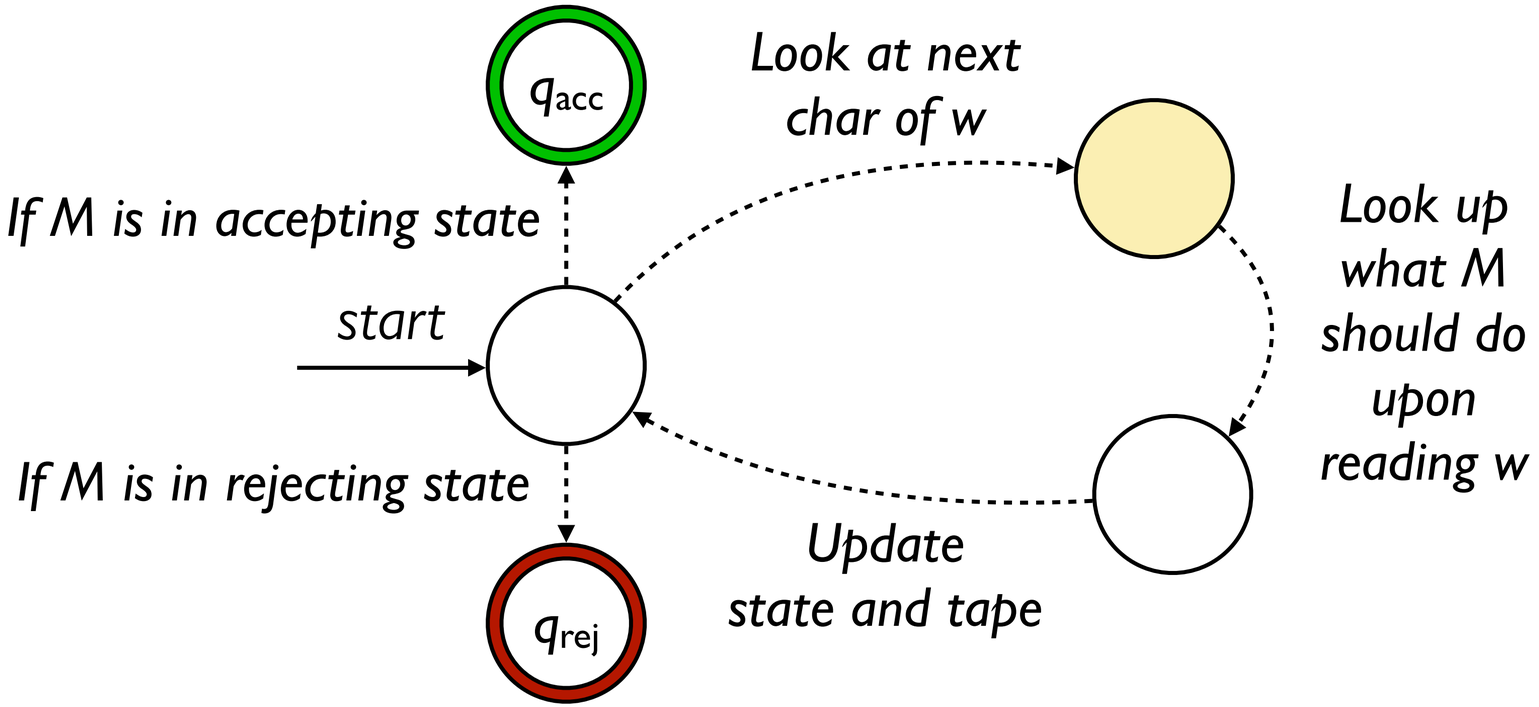
Input ⟨*M*, *w*⟩



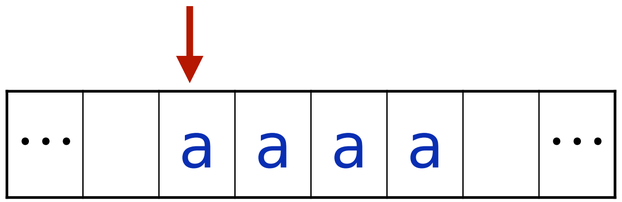
Machine *M*



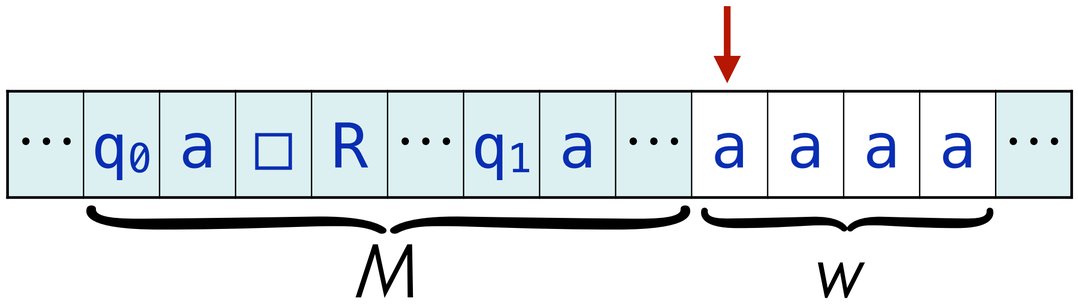
Universal TM *U*



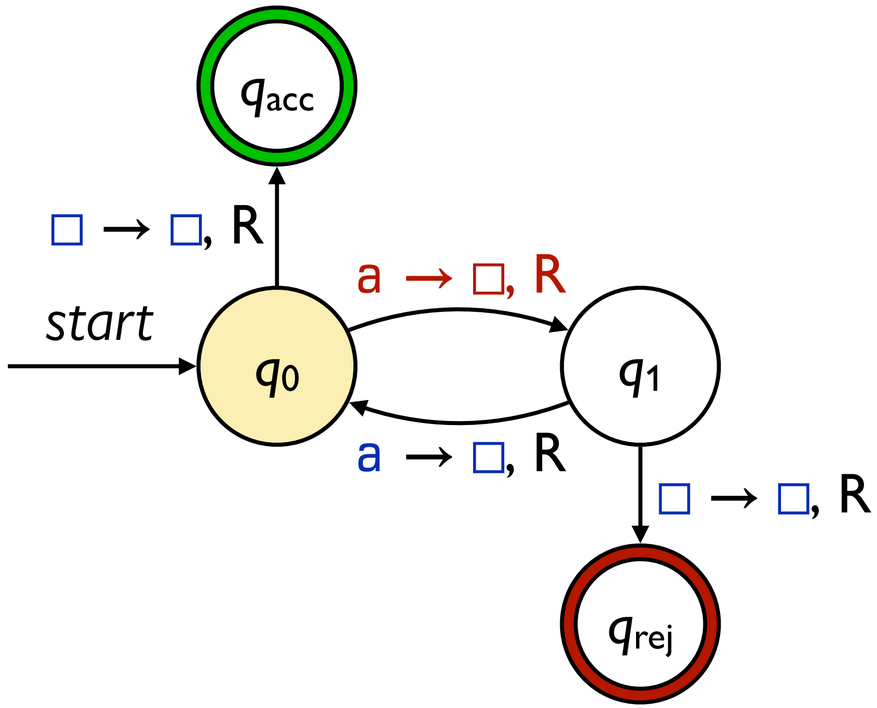
Input *w*



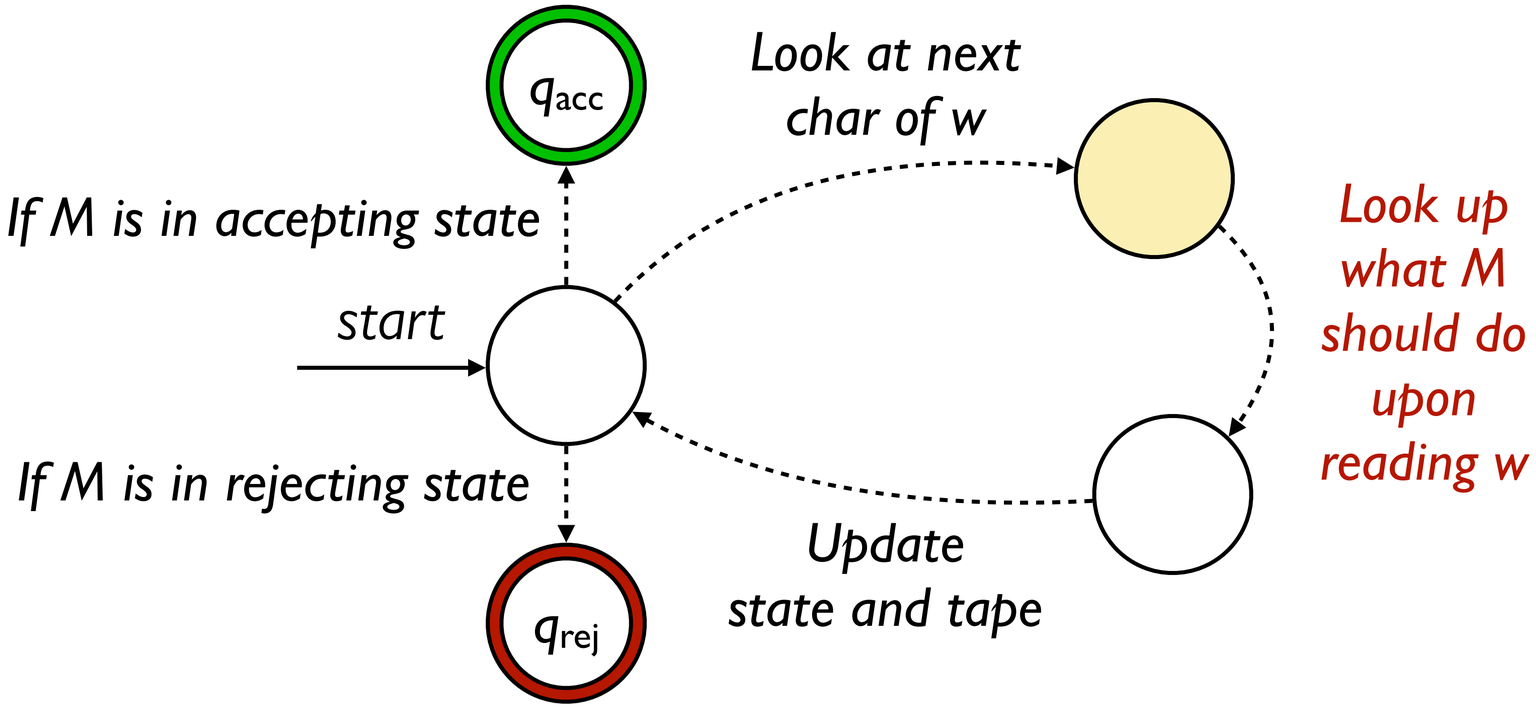
Input ⟨*M*, *w*⟩



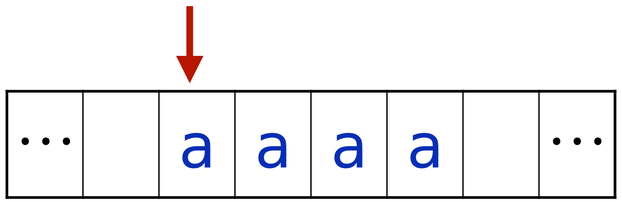
Machine *M*



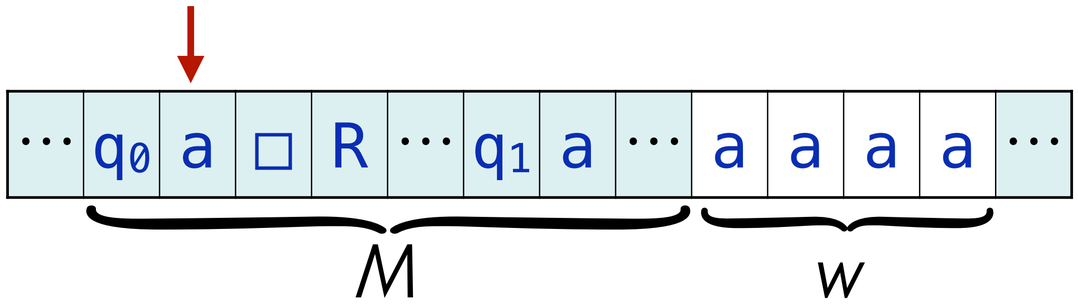
Universal TM *U*



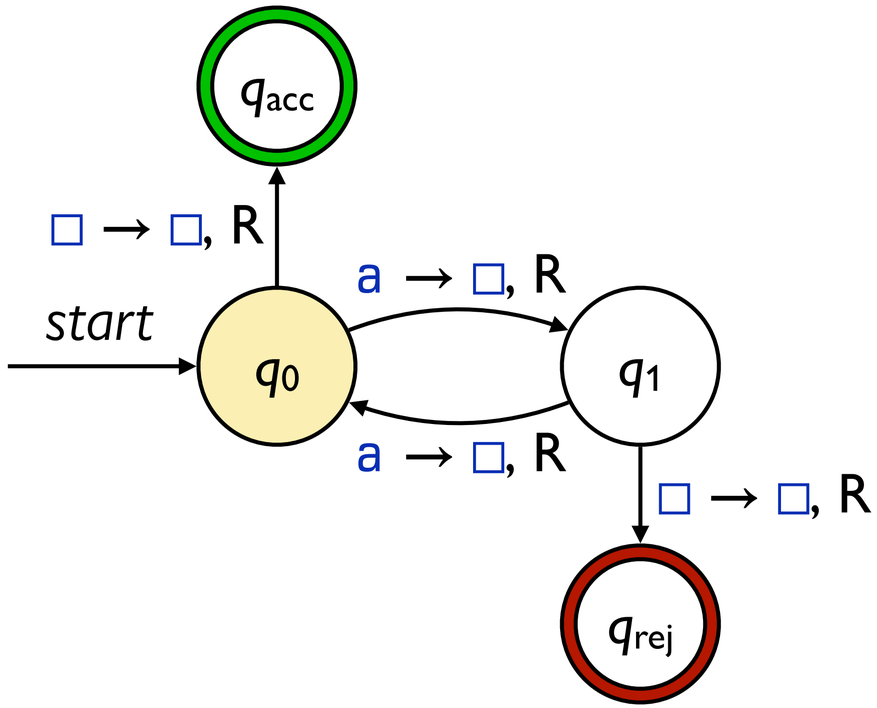
Input *w*



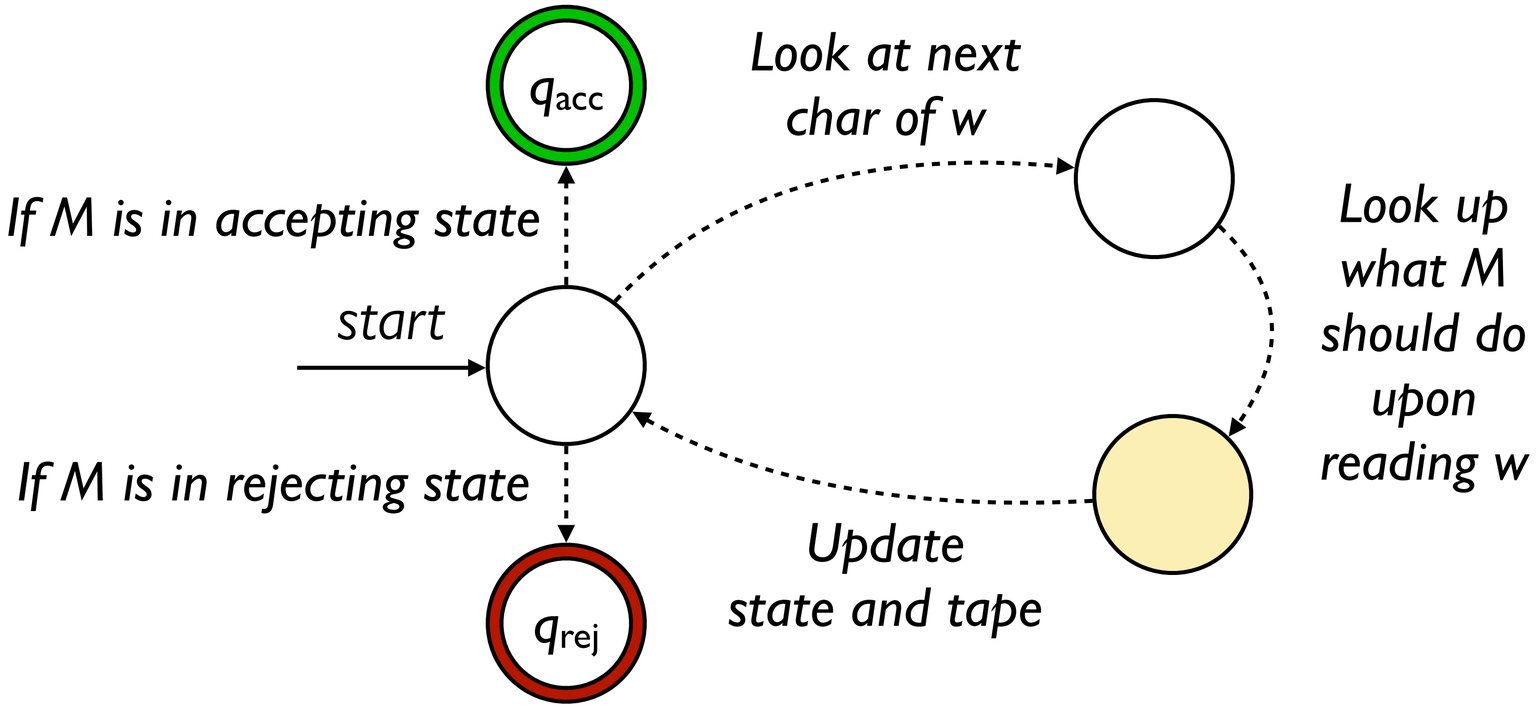
Input ⟨*M*, *w*⟩



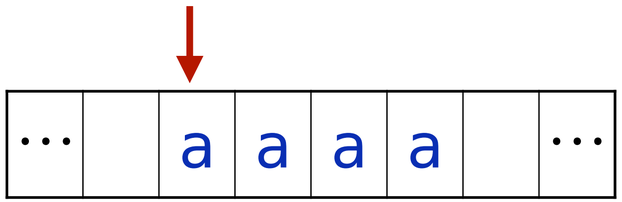
Machine *M*



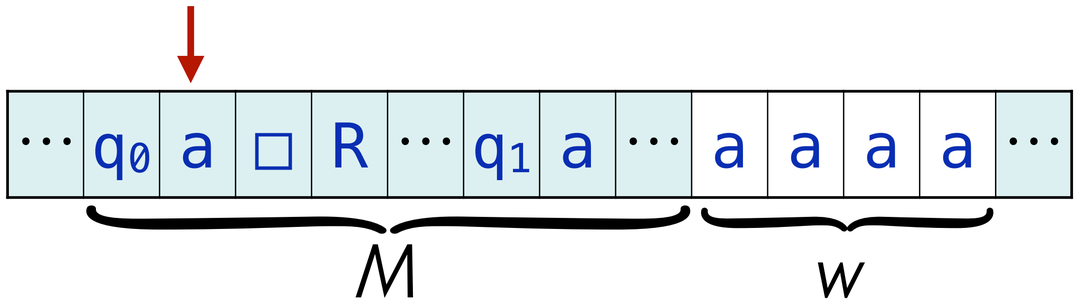
Universal TM *U*



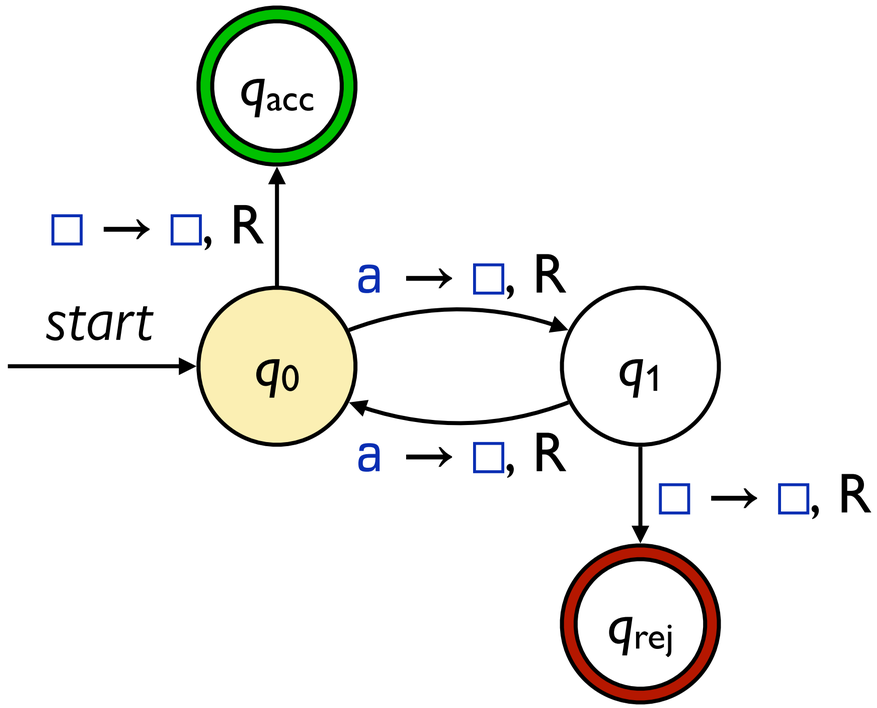
Input *w*



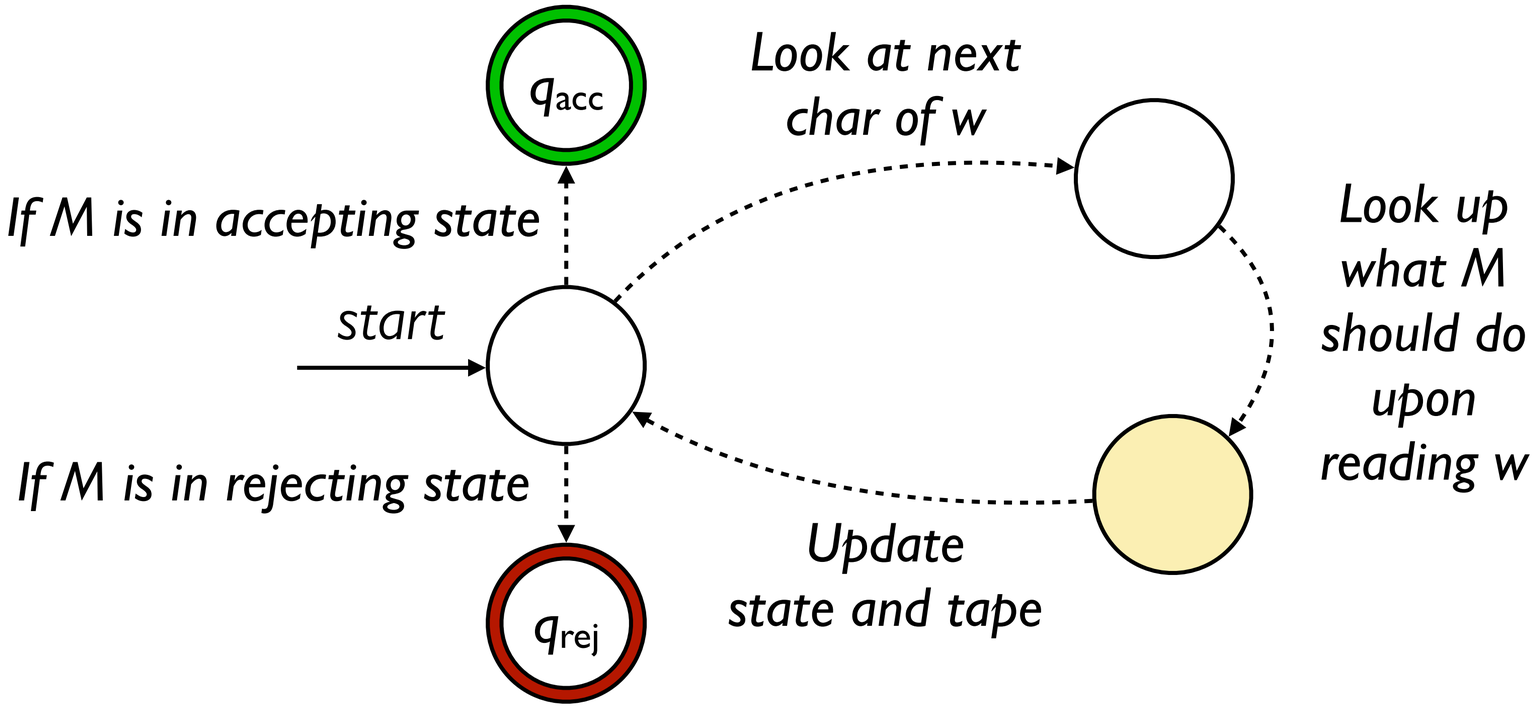
Input ⟨*M*, *w*⟩



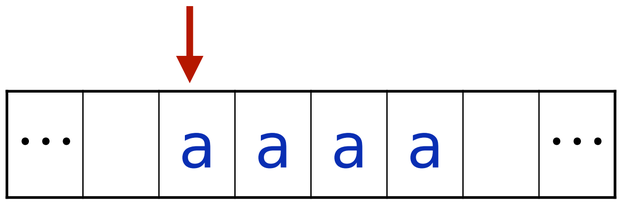
Machine *M*



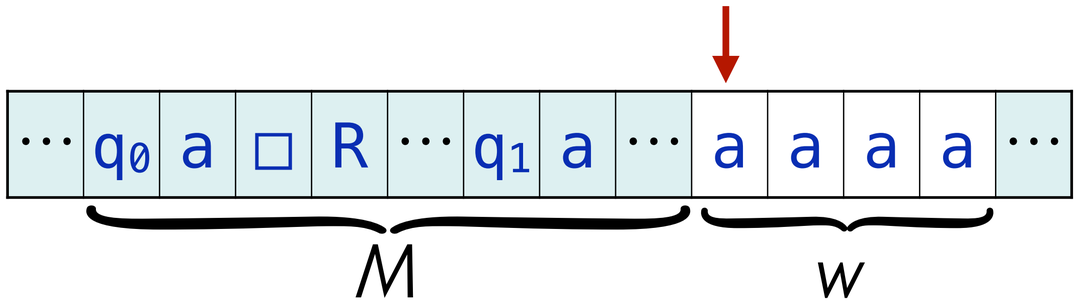
Universal TM *U*



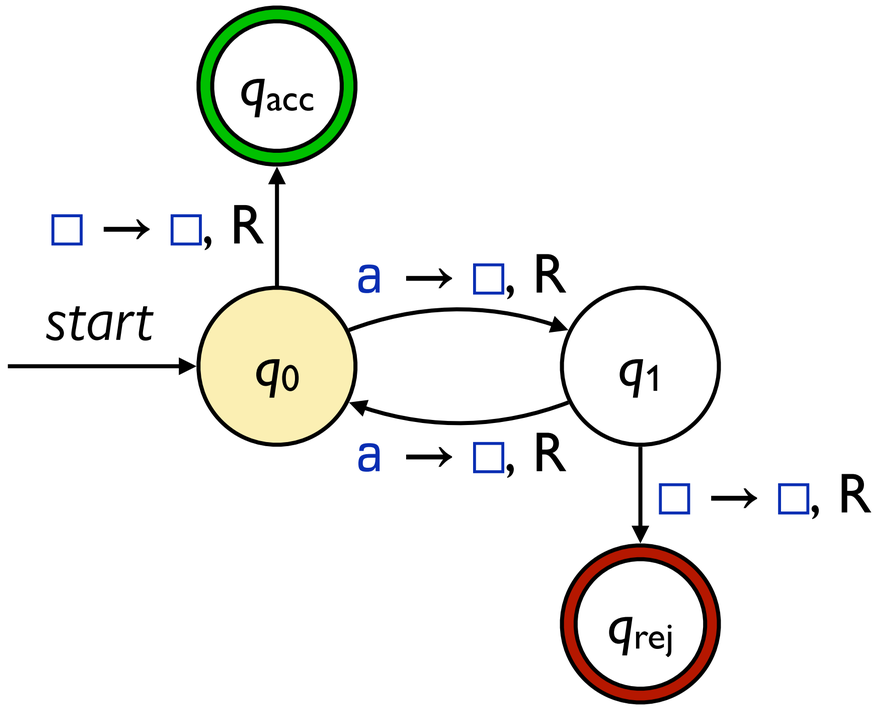
Input *w*



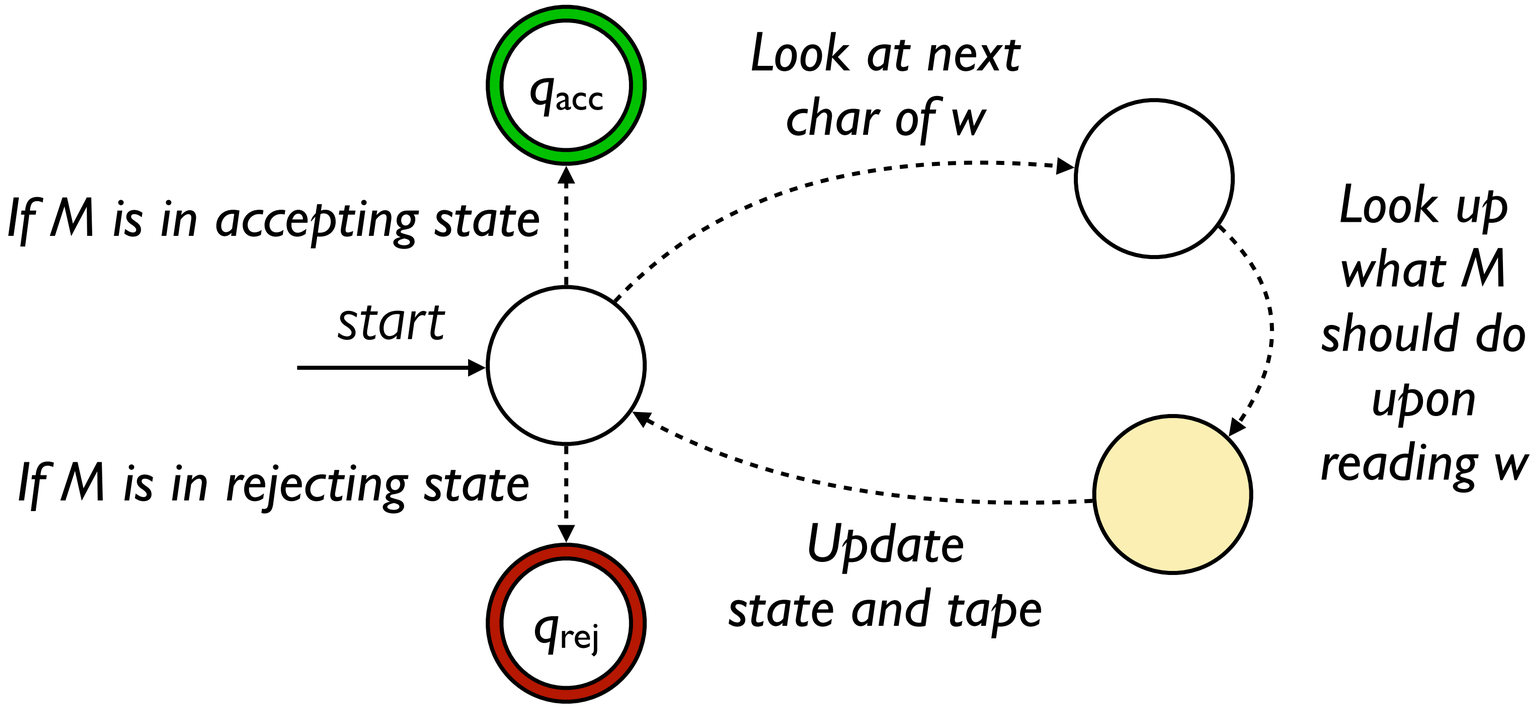
Input ⟨*M*, *w*⟩



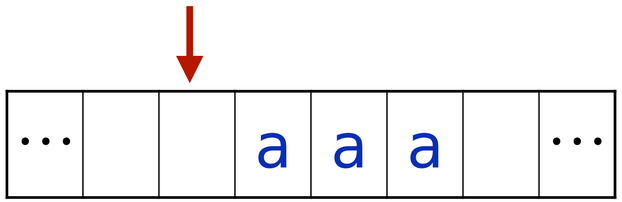
Machine *M*



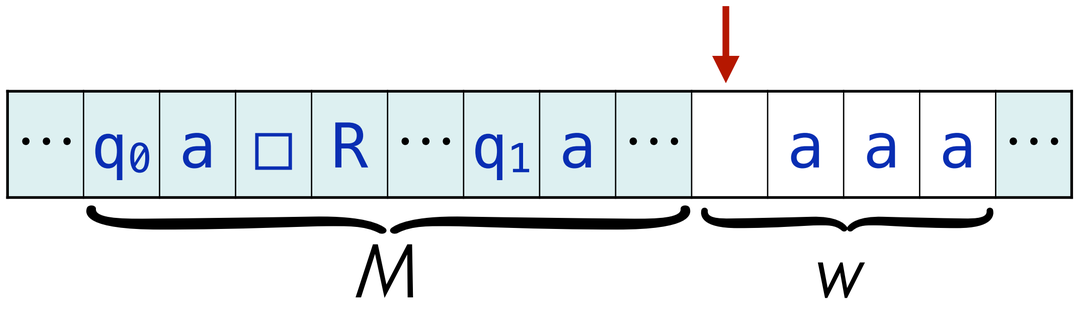
Universal TM *U*



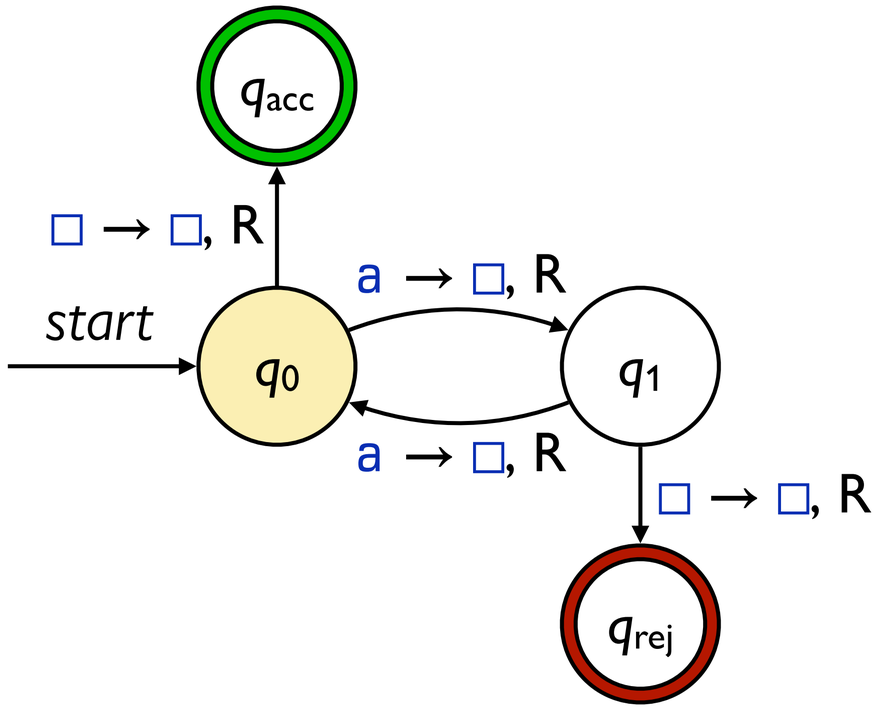
Input *w*



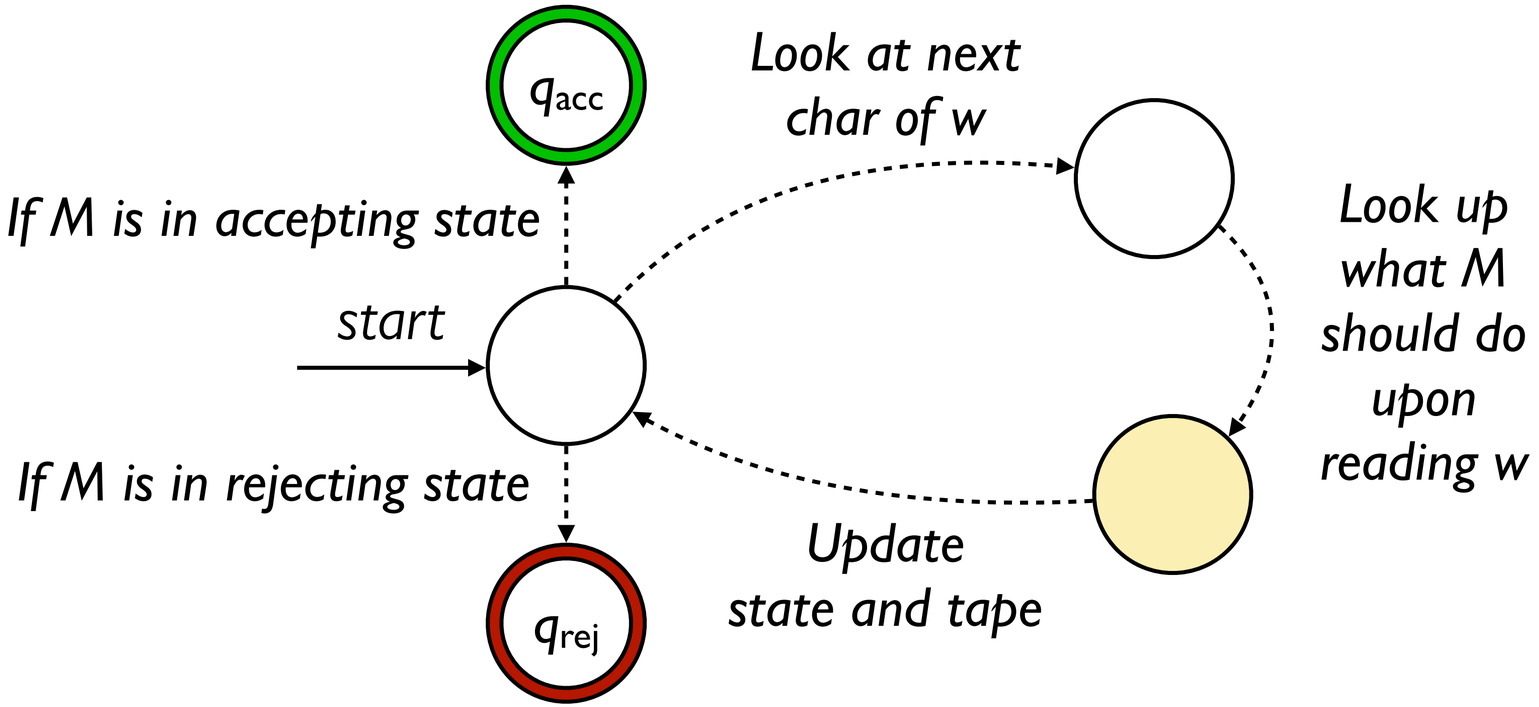
Input ⟨*M*, *w*⟩



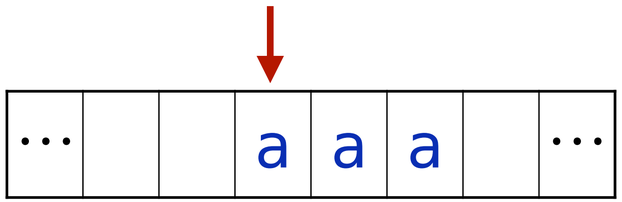
Machine *M*



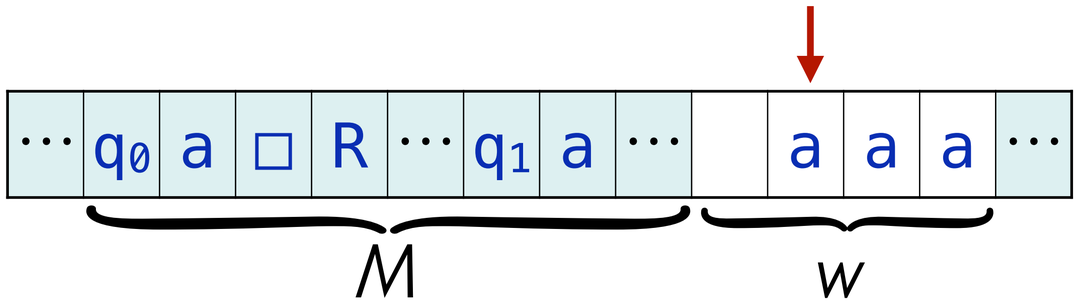
Universal TM *U*



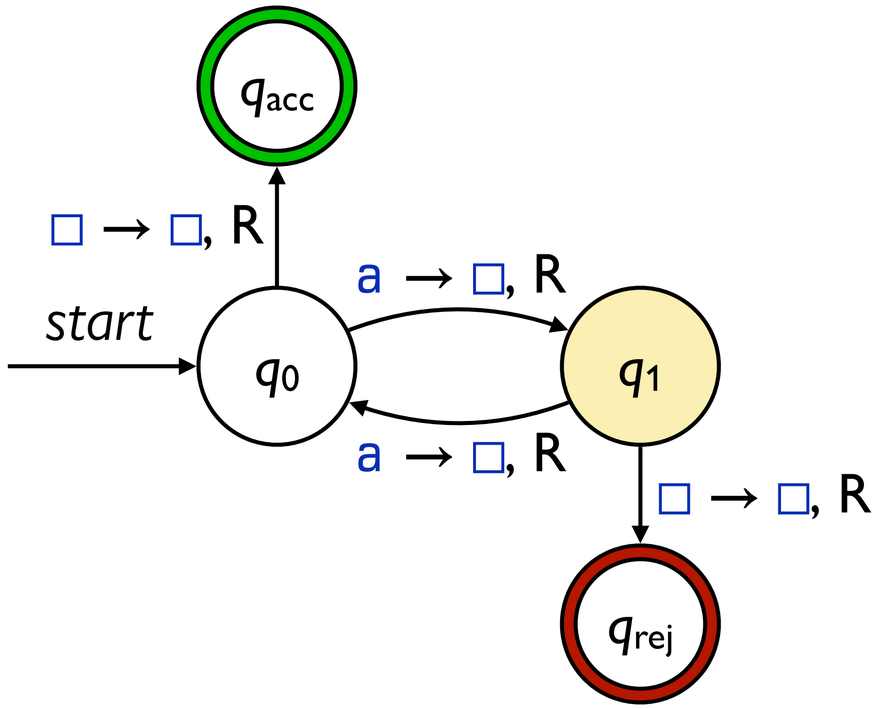
Input *w*



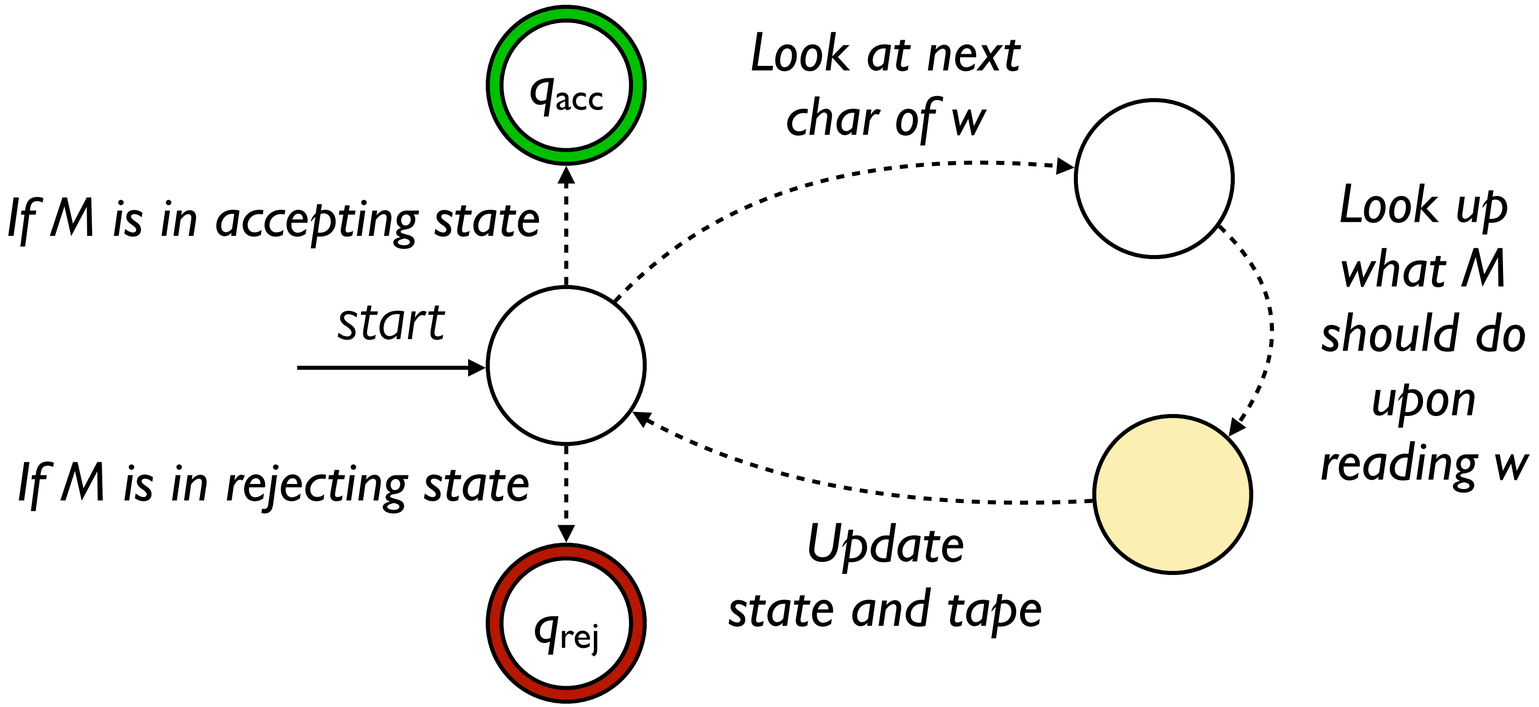
Input ⟨*M*, *w*⟩



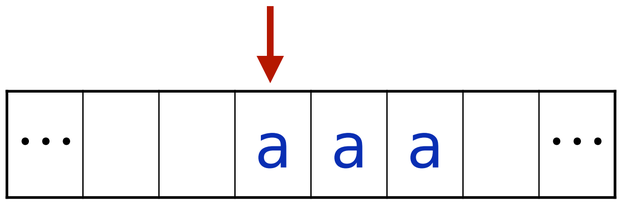
Machine *M*



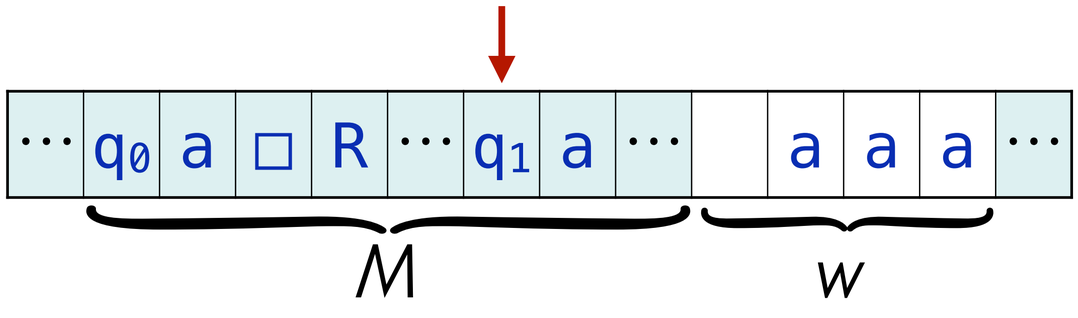
Universal TM *U*



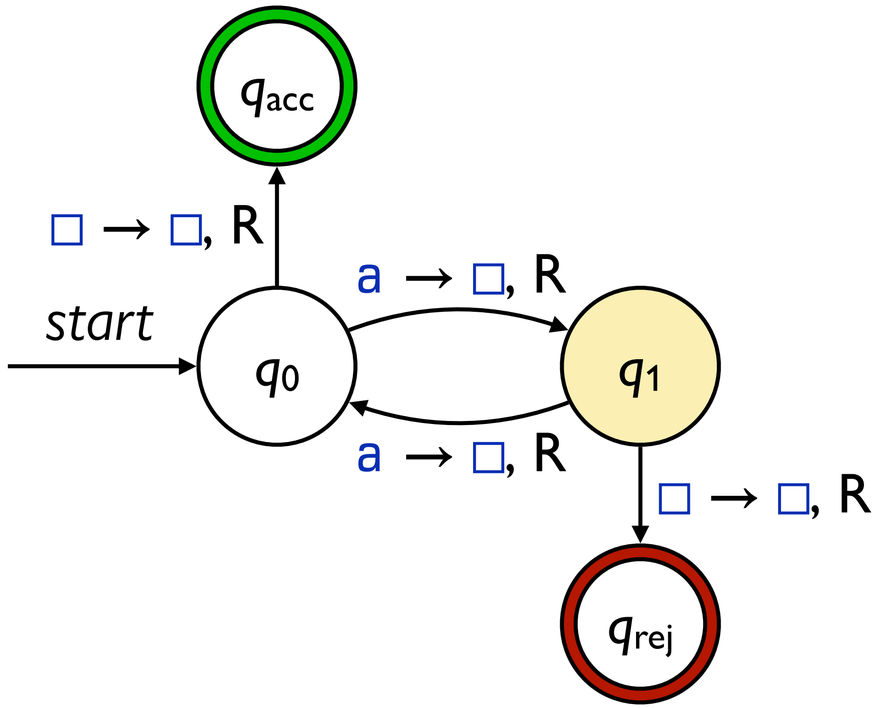
Input *w*



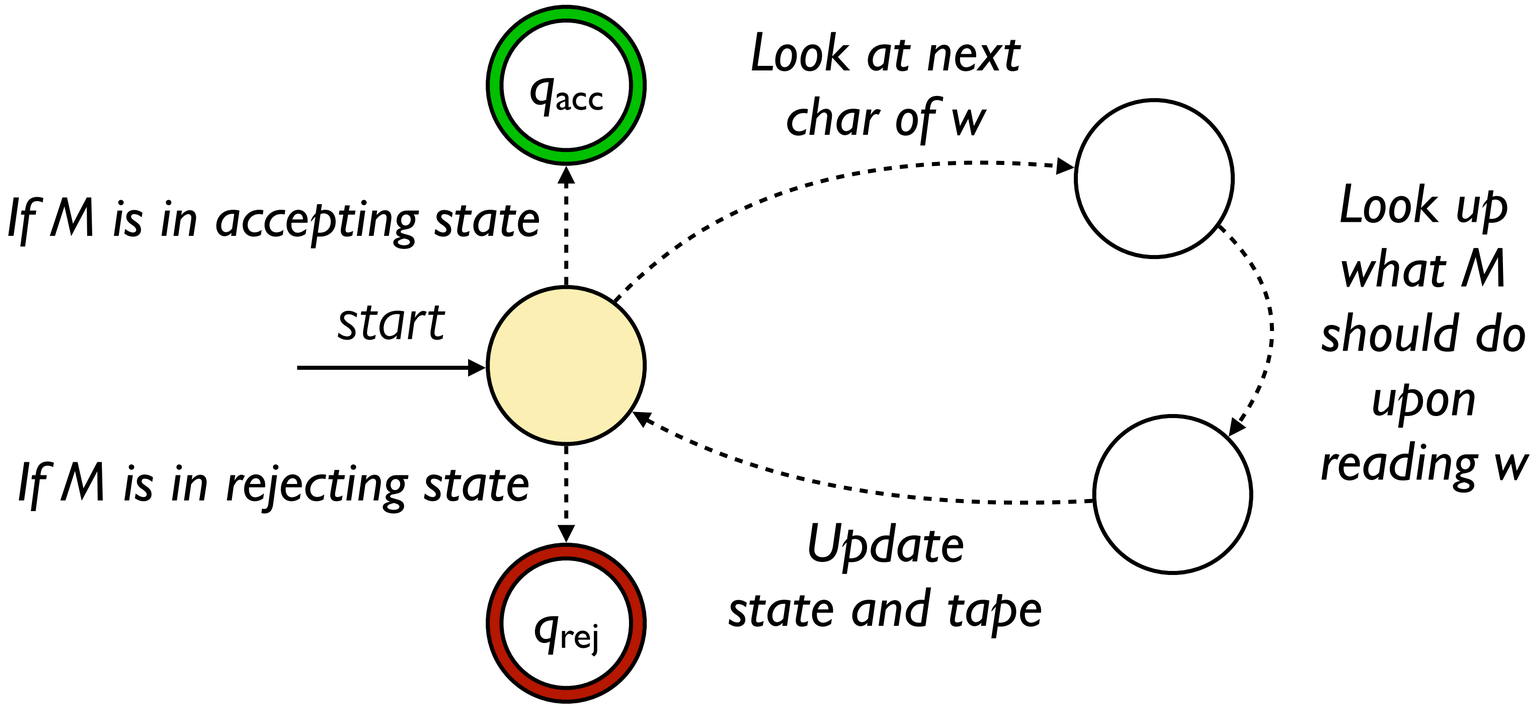
Input ⟨*M*, *w*⟩



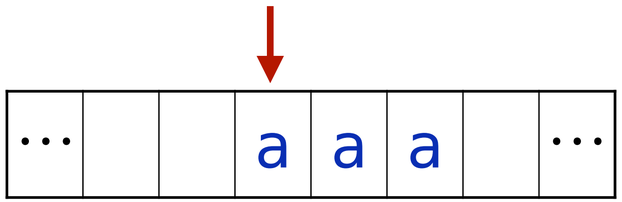
Machine *M*



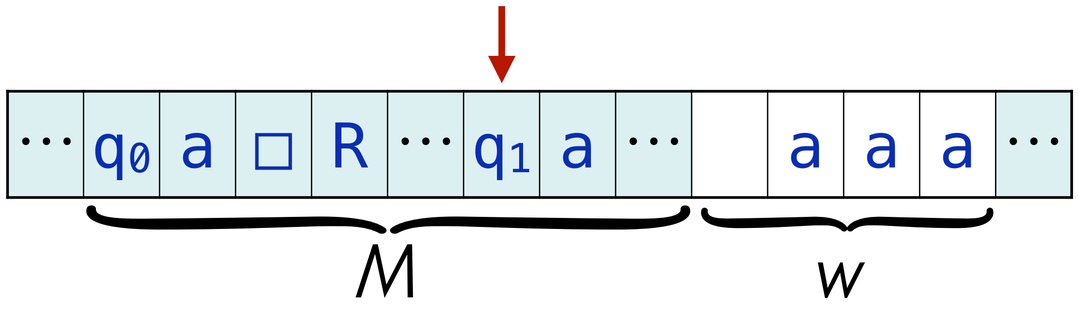
Universal TM *U*



Input *w*



Input ⟨*M*, *w*⟩



As a high-level description:

U = "On input $\langle M, w \rangle$, where M is a TM and $w \in \Sigma^*$,

Run M on w.

If M accepts w, U accepts $\langle M, w \rangle$.

If M rejects w, U rejects $\langle M, w \rangle$."

If M loops on w, then U loops as well.
This is implicit in the description.

The universal Turing machine *U* is the most powerful computational device that can be built.

- Assuming the Church–Turing thesis, any computation that can be performed by any computing system can be performed by a TM.
- The universal TM can "run" any TM, so it can perform any computation any TM can do.
- So *U* can do any computation that could ever be done by any possible feasible computing system!

And yet – it's just a simulator! All it does is simulate one step of a TM after another.

The language of *U*

U, when run on a string ⟨*M*, *w*⟩, where *M* is a TM and *w* is a string, will

accept ⟨*M*, w⟩ if *M* accepts *w,*

reject ⟨*M*, *w*⟩ if *M* rejects *w*, and

loop on ⟨*M*, *w*⟩ if *M* loops on *w*.

Although we didn't design *U* as a recognizer, it ***does*** recognize some language.

What language is that?

The language of *U*

U, when run on a string ⟨*M*, *w*⟩, where *M* is a TM and *w* is a string, will

accept ⟨*M*, w⟩ if *M* accepts *w*,

reject ⟨*M*, *w*⟩ if *M* rejects *w*, and

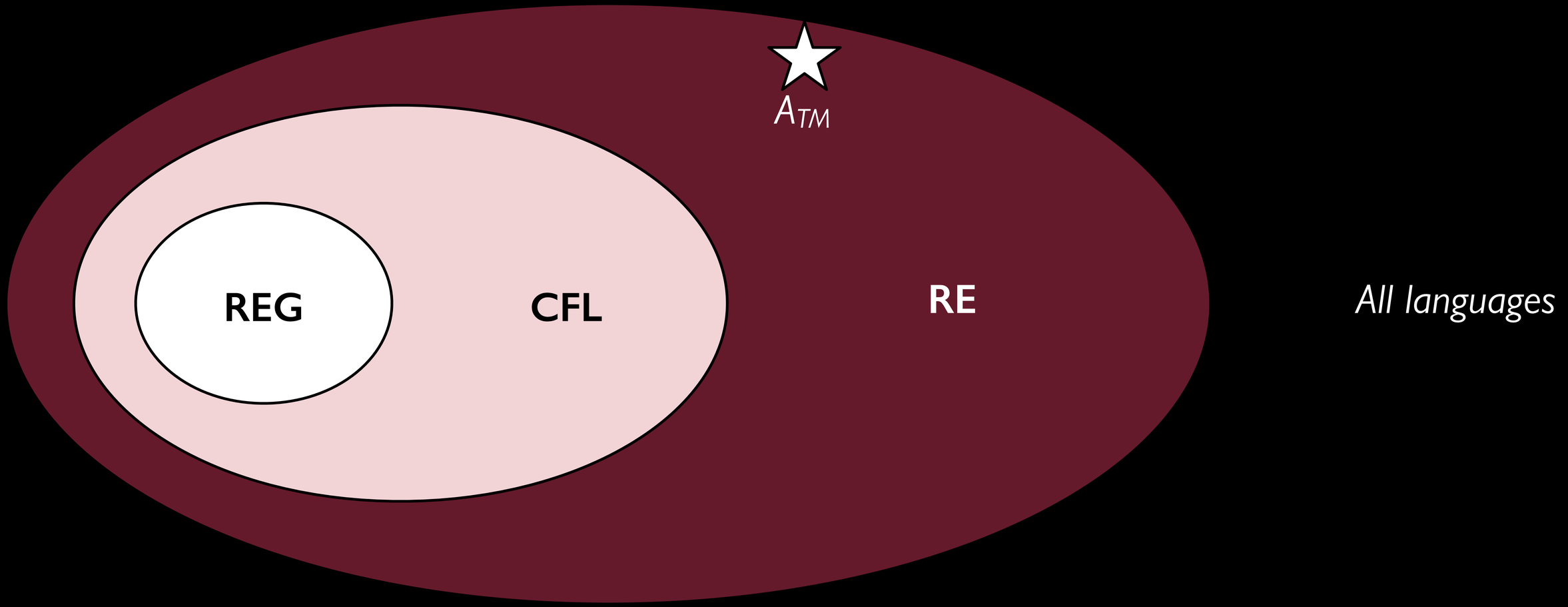
loop on ⟨*M*, *w*⟩ if *M* loops on *w*.

$$L(U) = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$
$$= \{\langle M, w\rangle \mid M \text{ is a TM and } w \in L(M)\}$$

The *acceptance language for Turing machines*, denoted A_{TM}, is the language of the universal Turing machine:

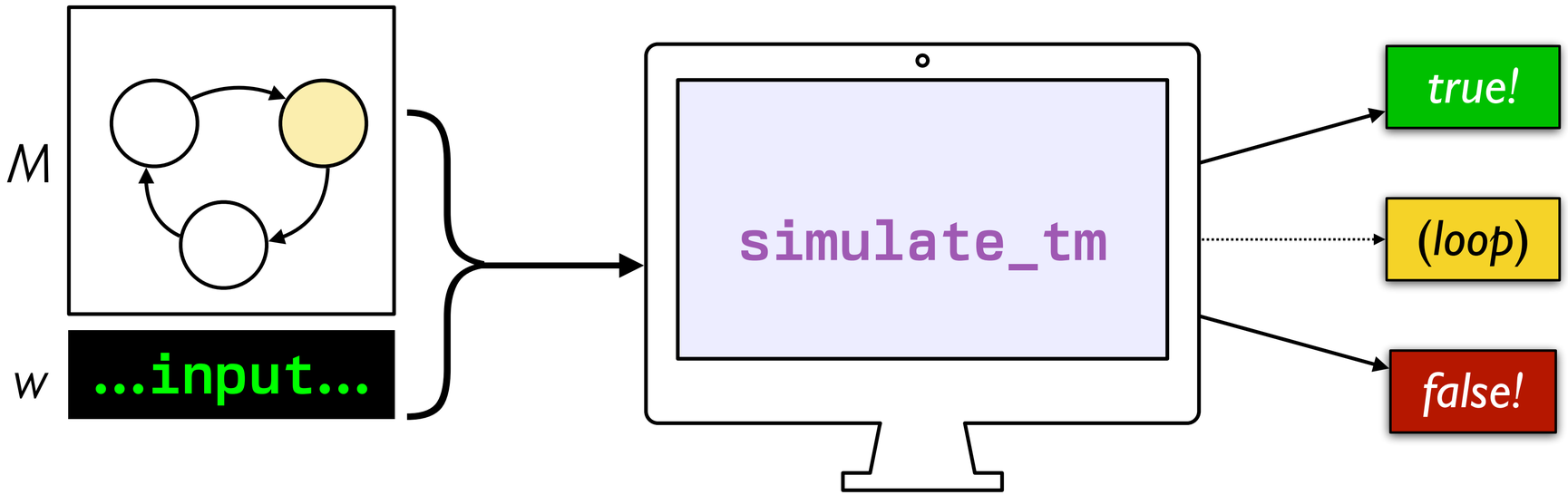
$$\begin{aligned} A_{TM} &= L(U) \\ &= \{\langle M, w\rangle \mid M \text{ is a TM and} \\ &\qquad\qquad M \text{ accepts } w\} \end{aligned}$$

Because there is a Turing machine, U, that recognizes A_{TM}, we know $A_{TM} \in$ **RE**.

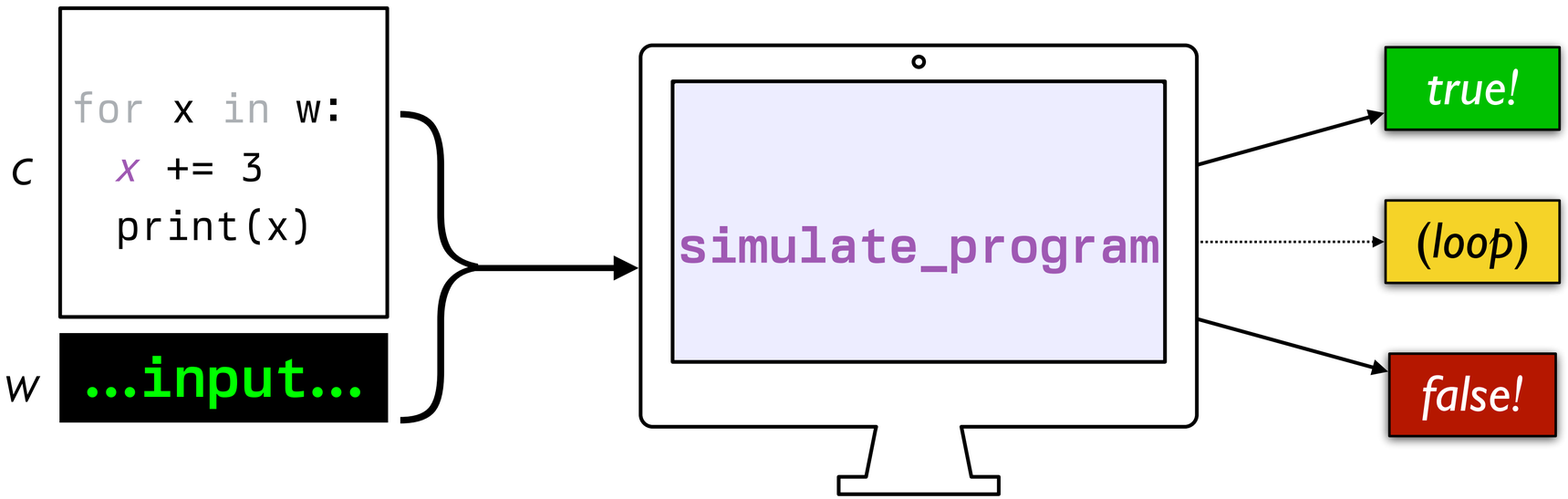

A_{TM}
REG
CFL
RE
All languages

Why do we care about universality?

Reason 1: It has practical consequences



What happens if we replace the Turing machine input with a normal computer program?



What happens if we replace the Turing machine input with a normal computer program?

An *interpreter* is a program that simulates other programs.

Python programs are usually executed by interpreters.

Your web browser interprets JavaScript code when it visits websites.

A *virtual machine* (or *emulator*) is a program that simulates an entire operating system.

Virtual machines are used in computer security, cloud computing, and even by individual end users.

Party like it's ~~1999~~ 1990

archive.org/details/msdos_Oregon_Trail_The_1990

It’s not a coincidence that interpreters and virtual machines are possible – Turing’s 1936 paper says that any general-purpose computing system must be able to do this!

The key idea behind the universal TM is that TMs can be fed as inputs to other TMs.

Similarly,

an interpreter is a program that takes other programs as inputs, and

an emulator is a program that takes entire computers as inputs.

This hits at the core idea that *computing devices can perform computations on other computing devices*.

Reason 2: It's philosophically interesting

Can computers think?

On 15 May 1951, Alan Turing delivered a radio lecture on the BBC, where he argued that

> "it is not altogether unreasonable to describe digital computers as brains".

Why would he think this, given the very limited abilities of computers of the time?

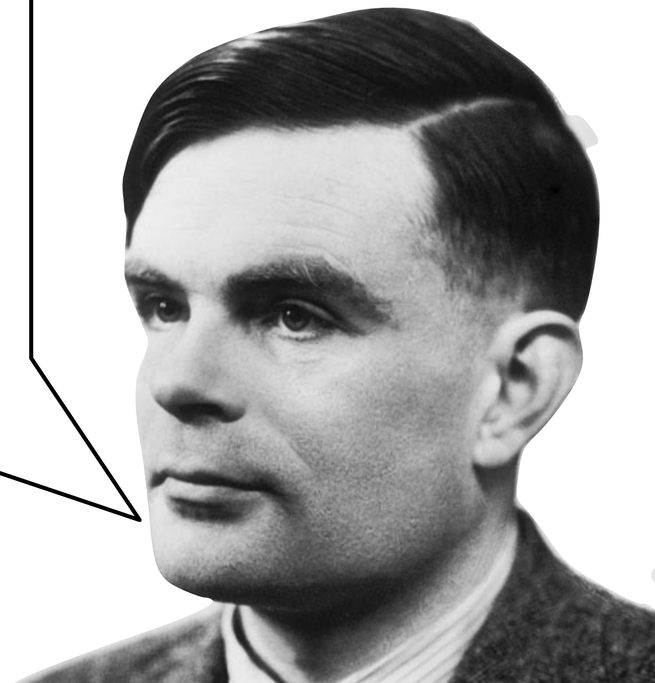
"I should also say that *'If **any** machine can be appropriately described as a brain, then any* [i.e., every] *digital computer can be so described.'*

"This last statement needs some explanation. It may appear rather startling, but with some reservations it appears to be an inescapable fact.

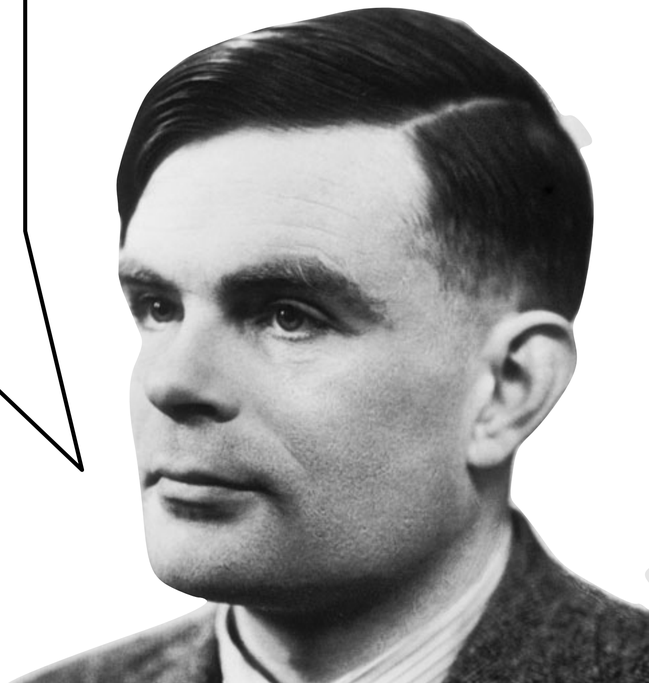
"It can be shown to follow from a characteristic property of digital computers, which I will call their *universality*…

"A digital computer is a *universal machine* in the sense that it can be made to replace any machine of a certain very wide class.

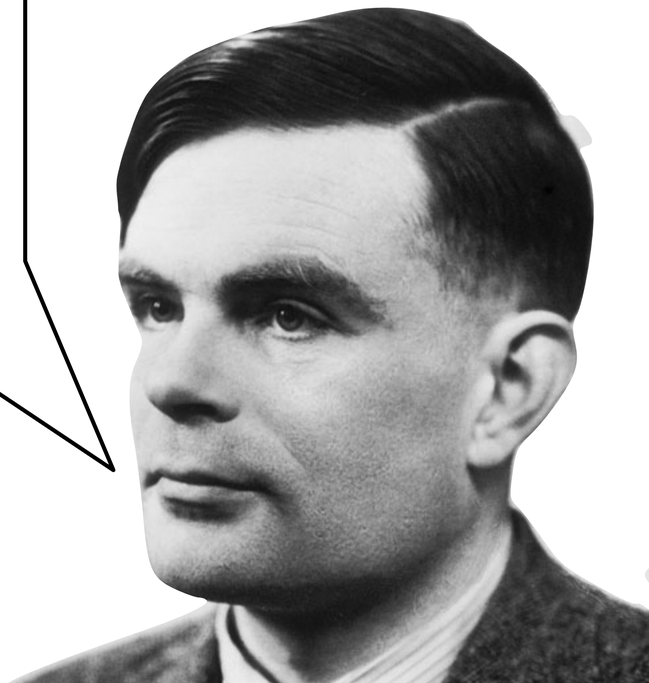
"It will not replace a bulldozer or a steam-engine or a telescope, but it will replace any rival design of calculating machine, that is to say any machine into which one can feed data and which will later print out results.



"In order to arrange for our computer to imitate a given machine it is only necessary to programme the computer to *calculate what the machine in question would do under given circumstances*, and in particular what answers it would print out. The computer can then be made to print out the same answers.

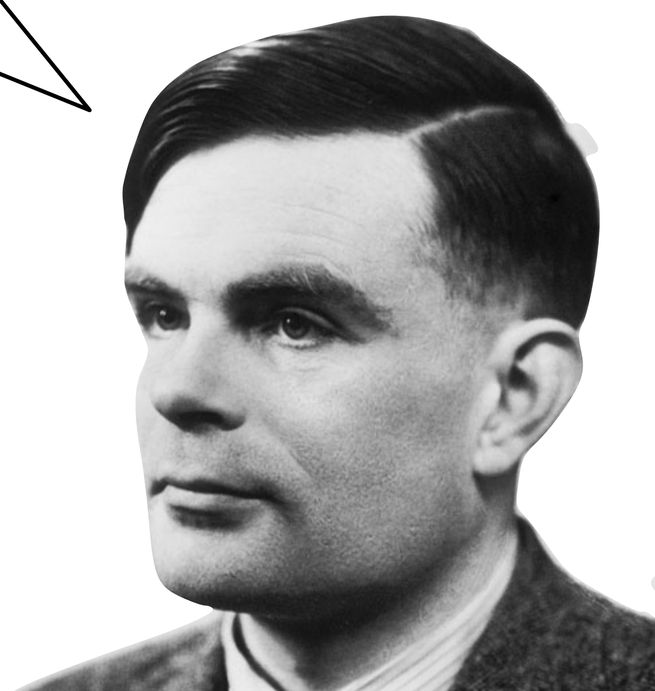


"If now some machine can be described as a brain *we have only to programme our digital computer to imitate it and it will also be a brain*. If it is accepted that real brains, as found in animals, and in particular in men, are a sort of machine it will follow that our digital computer suitably programmed, will behave like a brain."



"This argument involves several assumptions which can quite reasonably be challenged."

Alan Turing, 1951



Self-referentiality

We can write programs that act on themselves.

As a fun (but tricky) case, we can consider *quines* – programs that, when run, print out their own source code.

How would you write such a program?

selfie.py:

```
my_source = open("selfie.py").read()
print(my_source)
```

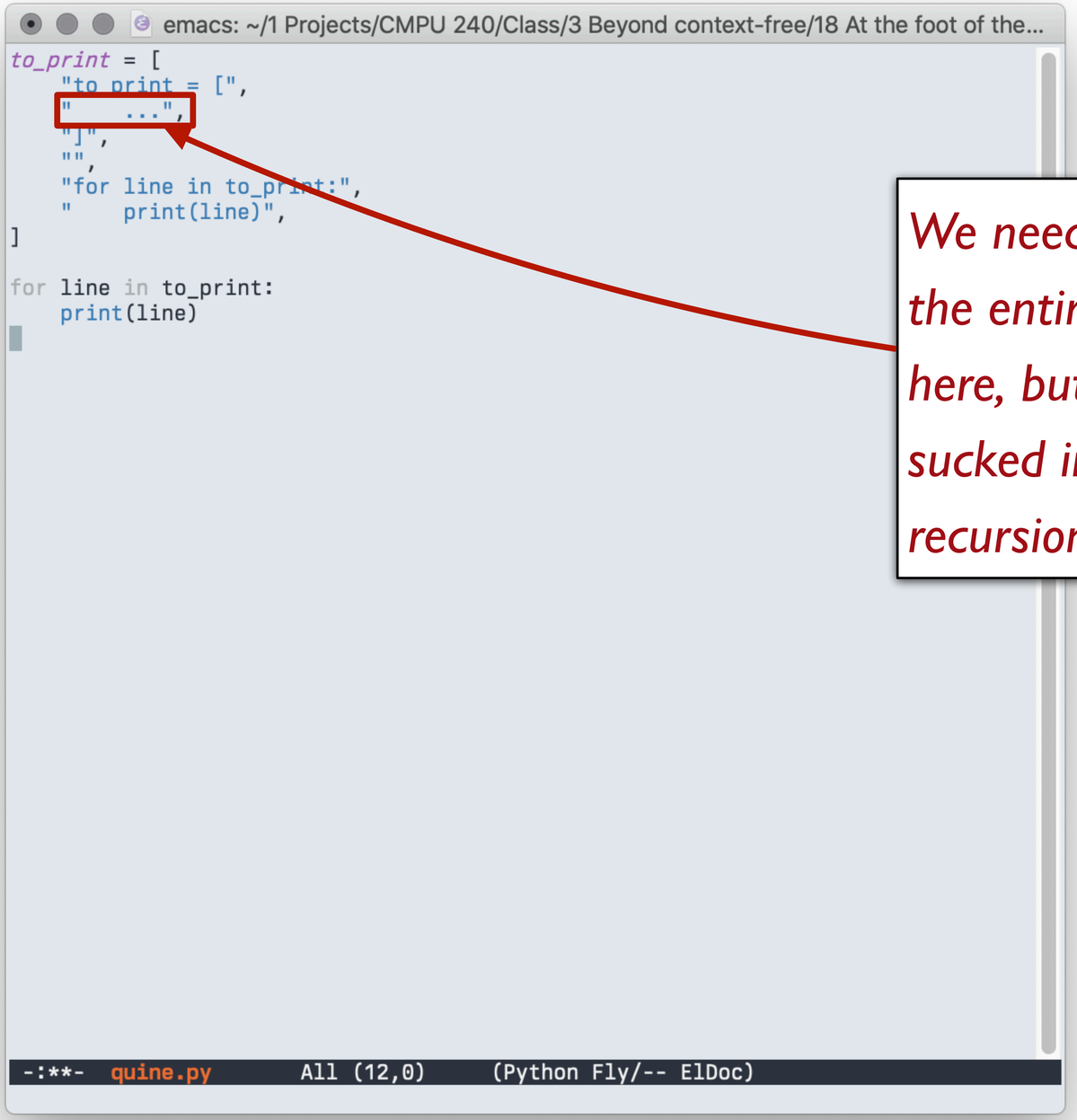
selfie.py:

```
my_source = open("selfie.py").read()
print(my_source)
```

It's cheating to read the source file – and what happens if someone changes the file after the program starts?

Coding break!


emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...
to_print = [
...
]
for line in to_print:
print(line)
-:**- quine.py All (7,0) (Python Fly/-- ElDoc)



We need to substitute the entire source code here, but without getting sucked into infinite recursion.


emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...
to_print = [
"to_print = [",
"@",
"]",
"",
"for line in to_print:",
"   print(line)",
]
for line in to_print:
if line == "@":
...
else:
print(line)
-:**- quine.py All (15,0) (Python Fly/-- ElDoc)


emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...
to_print = [
"to_print = [",
"@",
"]",
"",
"for line in to_print:",
"   print(line)",
]
def print_program_in_quotes():
...
for line in to_print:
if line == "@":
print_program_in_quotes()
else:
print(line)
-:--- quine.py All (18,0) (Python Fly/-- ElDoc)


emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...
to_print = [
"to_print = [",
"@",
"]",
"",
"for line in to_print:",
"   print(line)",
]
def print_program_in_quotes():
for line in to_print:
print("   " + line + ",")
for line in to_print:
if line == "@":
print_program_in_quotes()
else:
print(line)
-:--- quine.py All (19,0) (Python Fly/-- ElDoc)

emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...

```
to_print = [
    "to_print = [",
    "@",
    "]",
    "",
    "for line in to_print:",
    "    print(line)",
]

def print_program_in_quotes():
    for line in to_print:
        print("    " + line + ",")

for line in to_print:
    if line == "@":
        print_program_in_quotes()
    else:
        print(line)
```

-:--- quine.py All (19,0) (Pyth

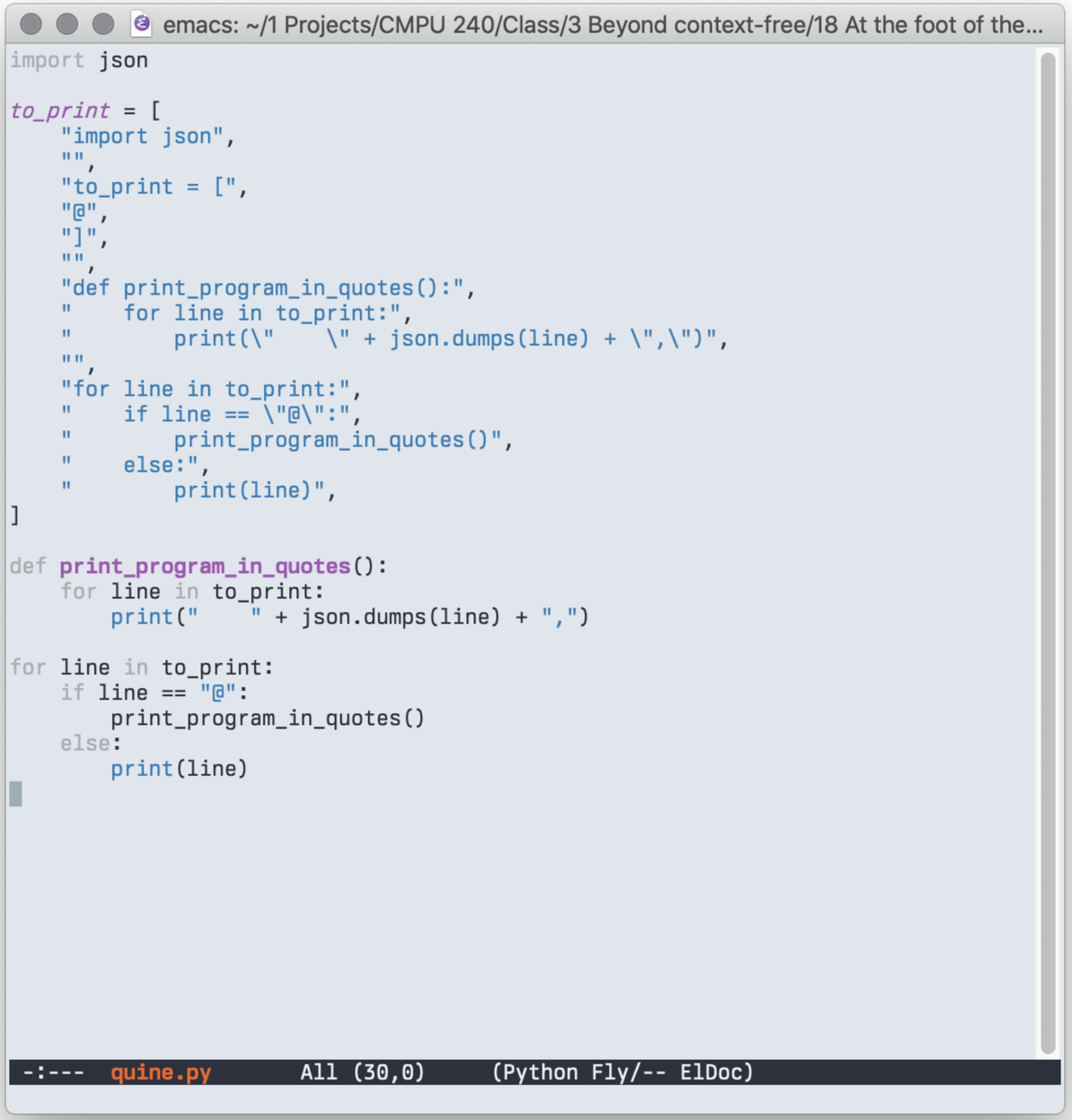
Close! But we need to quote the strings in **`to_print`** *and escape quotation marks inside them!*

jgordon@core: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot o...

```
; python3 quine.py
to_print = [
    to_print = [,
    @,
    ],
    ,
    def print_program_in_quotes():,
        for line in to_print:,
            print("    " + line + ","),
    ,
    for line in to_print:,
        if line == "@":,
            print_program_in_quotes(),
        else:,
            print(line),
]

def print_program_in_quotes():
    for line in to_print:
        print("    " + line + ",")

for line in to_print:
    if line == "@":
        print_program_in_quotes()
    else:
        print(line)
;
```



cs.vassar.edu/~cs240/class/quine.py

emacs: ~/1 Projects/CMPU 240/Class/3 Beyond context-free/18 At the foot of the...

```
import json

to_print = [
    "import json",
    "",
    "to_print = [",
    "@",
    "]",
    "",
    "def my_source():",
    "    result = \"\"",
    "    for line in to_print:",
    "        if line == \"@\":",
    "            for line in to_print:",
    "                result += json.dumps(line) + \"\\n\"",
    "        else:",
    "            result += line + \"\\n\"",
    "    return result",
    "",
    "if len(my_source()) % 2 == 0:",
    "    print(\"I am an even-length program.\")",
    "else:",
    "    print(\"I am an odd-length program.\")",
]

def my_source():
    result = ""
    for line in to_print:
        if line == "@":
            for line in to_print:
                result += json.dumps(line) + ",\n"
        else:
            result += line + "\n"
    return result

if len(my_source()) % 2 == 0:
    print("I am an even-length program.")
else:
    print("I am an odd-length program.")
```

-:--- obtain_source.py All (40,0) (Python Fly/-- ElDoc)

A self-referential program doesn't need to ***print*** *its source code; we can make it a string that we look at in order to compute!*

cs.vassar.edu/~cs240/class/obtain_source.py

The fact that we can write quines isn't a coincidence!

THEOREM It is possible to construct TMs that perform arbitrary computations on their own "source code".

This is (approximately) Kleene's second recursion theorem

In other words, any computing system that's equal to a Turing machine possess some mechanism for self-reference.

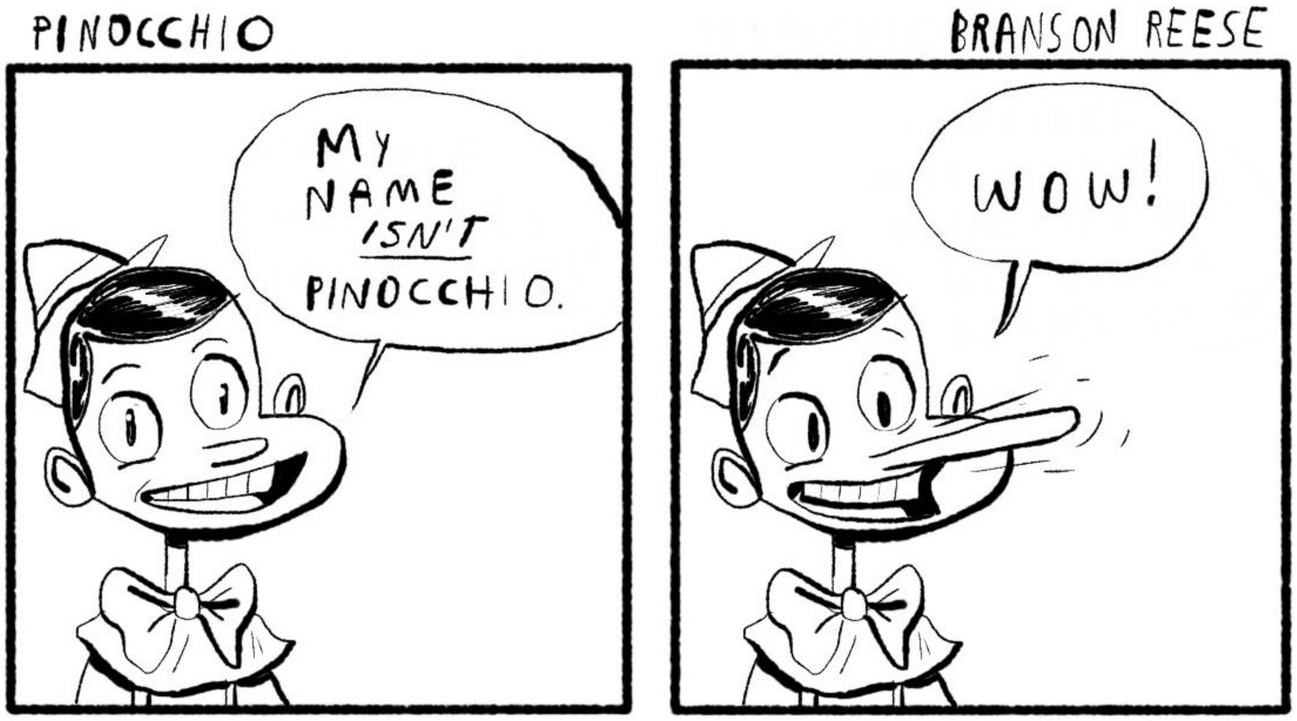
CLAIM Going forward, assume that any program can be augmented to include a **`my_source()`** function that returns a string representation of its source code.

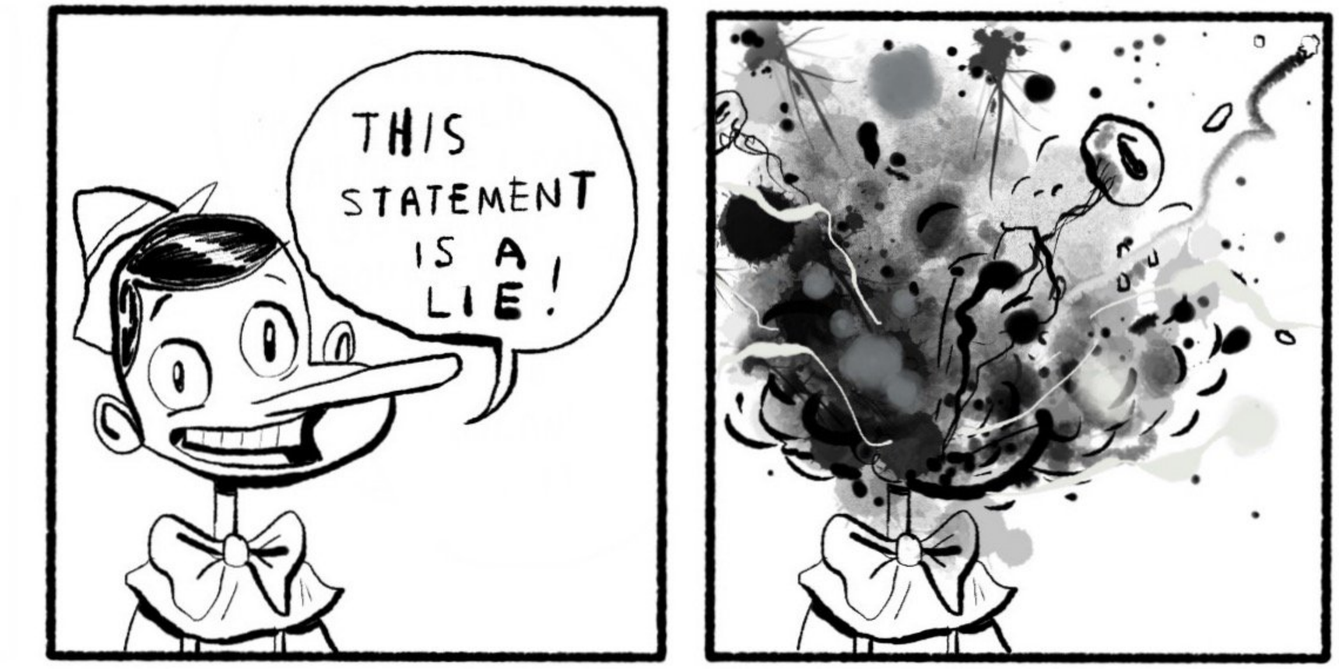
This means we can write programs like these:

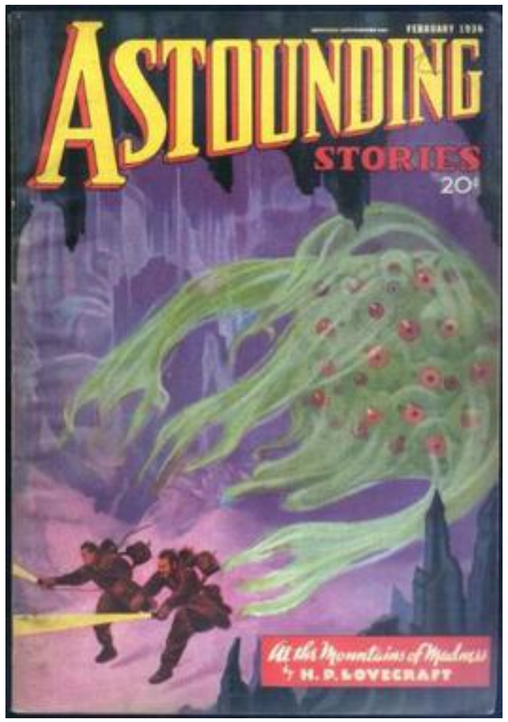
```
def narcissist(input: str) -> bool:
    me = my_source()
    return input == me
```

```
def accept_longer_strings(input: str) -> bool:
    me = my_source()
    return len(input) > len(me)
```

Teaser: Self-reference lets machines compute on themselves. That will let them do *cruel and unusual things*.


PINOCCHIO
MY NAME ISN'T PINOCCHIO.
BRANSON REESE
WOW!


THIS STATEMENT IS A LIE!



We now stand at the foot of the mountains of undecidability.

Next time we'll see how the emergent properties of universality and self-reference bring us to the limits of computation!

Acknowledgments

This lecture incorporates material from:

Amy Liu, Stanford University

Keith Schwarz, Stanford University

Michael Sipser, *Introduction to the Theory of Computation*