Try the following problems so we can discuss any questions or difficulties you have, either during the review session or during my office hours before the exam. The problems won’t be collected or graded. Collaboration is encouraged; please discuss these problems with classmates to help each other. After the review session, I’ll release example solutions.

Problem 1

Provided below is a Turing machine for the language \( \{a^n b^n \mid n \in \mathbb{N}_0\} \). Note that \( B \) is being used for a blank, and the input alphabet is \( \{a, b\} \).

Modify the above to create a TM for the language \( \{a^n b^{2n} \mid n \in \mathbb{N}_0\} \), and draw a diagram for the resulting TM.
Problem 2

Each of the "proofs" below claims to prove a conjecture. The conjectures might be true or false, but both of the proofs are bad. For each proof, explain clearly why the provided proof does not satisfactorily prove the given conjecture.

a. **Conjecture**: The “BusyBee” language, defined below, is undecidable:

\[ L_{\text{BusyBee}} = \{ \langle M, w, k \rangle \}, \]
where \( M \) describes a Turing machine and \( k \) is the number of different states \( M \) enters before halting on input \( w \). (Note that \( q_{\text{accept}} \) and \( q_{\text{reject}} \) are counted toward this total.)

**Proof by reduction**: We prove that \( L_{\text{BusyBee}} \) is undecidable by reducing \( \text{HALT}_{TM} \) to it. Construct \( M_{\text{BusyBee}} \) given a machine \( M_{\text{HALT}} \) that decides \( \text{HALT}_{TM} \).

\[ M_{\text{BusyBee}} = \]

Simulate \( M_{\text{HALT}} \) on input \( \langle M, w \rangle \). If \( M_{\text{HALT}} \) accepts, simulate \( M \) on \( w \). On a second tape, list the states of \( M \) and add a mark on each state that is entered. Count the number of marked states on the second tape. If the number matches \( k \), accept. Otherwise, reject (without simulating \( M \) on \( w \) since it does not halt). Since building \( M_{\text{BusyBee}} \) requires \( M_{\text{HALT}} \), which we know does not exist, this proves that \( L_{\text{BusyBee}} \) is undecidable.

Your answer should identify a **fundamental flaw** in the proof (for example, by explaining what it actually proves instead of the conjecture), not a minor issue like lack of sufficient detail.
b. **Conjecture**: The set of languages that can be recognized by a deterministic pushdown automaton is equivalent to the set of languages that can be recognized by a Turing machine.

**Proof by simulation**: We prove the conjecture by showing how to simulate any DPDA with a TM…(*unnecessary details elided*)…Thus, we can simulate a DPDA with a TM. This proves a DPDA is equivalent to a TM.
Problem 3

Consider the Turing machine \( M = (\{ q_0, q_1, q_{\text{accept}}, q_{\text{reject}} \}, \{ 0, 1 \}, \{ \square \}, \delta, q_0, q_{\text{accept}}, q_{\text{reject}}) \), where \( \delta \) consists of the following rules:

\[
\begin{align*}
\delta(q_0, \square) &= (q_{\text{reject}}, \square, R) \\
\delta(q_0, 0) &= (q_0, \square, R) \\
\delta(q_0, 1) &= (q_1, \square, R) \\
\delta(q_1, \square) &= (q_{\text{accept}}, \square, R) \\
\delta(q_1, 0) &= (q_{\text{reject}}, \square, R) \\
\delta(q_1, 1) &= (q_1, \square, R)
\end{align*}
\]

a. Informally but clearly describe the language \( L(M) \).

b. Give a string in this language that is of length at least 4, and show the sequence of moves of this TM when this string is the input to \( M \).
Problem 4

Answer each of the following as indicated.

a. Indicate whether the following is true or false and fully justify your answer: If language $A$ is a subset of language $B$, and $A$ is Turing-recognizable, then $B$ is Turing-recognizable.

b. Indicate whether the following is true or false and fully justify your answer: A nondeterministic TM accepts a string $w$ if and only if every possible computation on the input leads to an accepting state.

c. Indicate whether the following is true or false and fully justify your answer: The computational path of a TM on an input $w$ always either halts accepting, halts rejecting, or enters a never-halting loop.

d. Circle the number to the left of each of the languages below that is Turing-decidable but not context-free:
   1. $\{ww^R \mid w \in \{a,b\}^*\}$
   2. $\{wcw \mid w \in \{a,b\}^*\}$
   3. $\{a^{2n}b^n c^{2n} \mid n \in \mathbb{N}_0\}$

e. Circle the number to the left of each of the languages below that is Turing-recognizable but not Turing-decidable:
   1. $\{\langle M, w \rangle \mid \text{TM } M \text{ accepts input } w\}$
   2. $\{\langle M \rangle \mid L(M) \neq \varnothing\}$
   3. $\{\langle M_1, M_2 \rangle \mid L(M_1) = L(M_2)\}$

f. Circle the number to the left of each of the languages below that is not Turing-recognizable:
   1. $\{\langle M, w \rangle \mid \text{TM } M \text{ halts on input } w\}$
   2. $\{\langle M \rangle \mid L(M) \neq \varnothing\}$
   3. $\{\langle M_1, M_2 \rangle \mid L(M_1) \cap L(M_2) = \varnothing\}$
Problem 5

Determine whether the following problems are decidable or not:

a. Is a number $m$ prime?

b. Given a regular language $L$ and a string $w$, is $w \in L$?
Problem 6

Prove that the language $L = \{ \langle M \rangle \mid M$ is a TM and $M$ accepts string $w$ whenever it accepts $w^R \}$ is undecidable.