Broken Pumping Lemma proofs

The following slides show several applications of the Pumping Lemma to show that the language over $\Sigma=\{a,b\}^*$ consisting of strings that are palindromes is not regular.

Each proof has a flaw that makes it inadequate or incorrect.

Broken proof 1

Suppose that the set of palindromes were regular. Let $n$ be the value from the Pumping Lemma. Consider the string $w = 0^n110^n$. $w$ is clearly a palindrome and $|w| \geq n$. By the Pumping Lemma, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the Pumping Lemma.

So, pick any $x$, $y$, and $z$ such that $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$.

Because $|xy| \leq n$, $xy$ is entirely contained in the $0^n$ at the start of $w$. So $x$ and $y$ consist entirely of zeros.

Now, consider $xy^2z$. By the Pumping Lemma, $xy^2z$ must be in the language. But $xy^2z$ can’t be a palindrome. This means that the set of palindromes doesn’t satisfy the Pumping Lemma and, thus, the set of palindromes cannot be regular.

This proof doesn’t contain enough detail about why $xy^2z$ isn’t a palindrome.

Broken proof 2

Suppose that the set of palindromes were regular. Let $n$ be the value from the Pumping Lemma. Consider the string $w = 00011000$. $w$ is clearly a palindrome. By the Pumping Lemma, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the Pumping Lemma.

So, pick any $x$, $y$, and $z$ such that $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$.

Because $|xy| \leq n$, $xy$ is entirely contained in the $0^n$ at the start of $w$. So, $x$ and $y$ consist entirely of zeros.

Now, consider $xy^2z$. By the Pumping Lemma, $xy^2z$ must be in the language. But $xy^2z$ can’t be a palindrome. This means that the set of palindromes doesn’t satisfy the Pumping Lemma and, thus, the set of palindromes cannot be regular.

This proof has an even bigger mistake. It’s using a fixed string $w$, whose length doesn’t depend on $n$. There’s no way to flesh out the later bits of the proof if you have done this. The string $w$ has to get longer as $n$ gets bigger.
Broken proof 3

Suppose that the set of palindromes were regular. Let $n$ be the value from the Pumping Lemma. Consider a string $w$ in $L$, with $|w| \geq n$. By the Pumping Lemma, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the Pumping Lemma.

So, pick any $x$, $y$, and $z$ such that $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$. Because $|xy| \leq n$, $xy$ is entirely contained in the $(01)^n$ at the start of $w$. So $y$ must be $(01)^i$ for some integer $i \geq 1$.

Now, consider $xy^2z$. By the Pumping Lemma, $xy^2z$ must be in the language. But $xy^2z$ can’t be a palindrome. This means that the set of palindromes doesn’t satisfy the Pumping Lemma and, thus, the set of palindromes cannot be regular.

Wow – this proof never assigned a value to $w$! There’s no way you can write a Pumping Lemma proof without a specific value for $w$.

A correct solution

Suppose that the set of palindromes were regular. Let $n$ be the value from the Pumping Lemma. Consider the string $w = 0^n110^n$, $w$ is clearly a palindrome and $|w| \geq n$. By the Pumping Lemma, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the Pumping Lemma.

So, pick any $x$, $y$, and $z$ such that $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$. Because $|xy| \leq n$, $xy$ is entirely contained in the $0^n$ at the start of $w$. So $x$ and $y$ consist entirely of zeros, i.e., $x = 0^i$ and $y = 0^j$. Then $z$ must look like $0^{i+k}10^n$, where $i + j + k = n$.

Now, consider $xy^2z$. By the Pumping Lemma, $xy^2z$ must be in the language. But $xy^2z = 0^{i+k}110^n$. This is just $0^{i+k+1}0^n$. Since $|y| \geq 1$, we know that $j \geq 1$. So $i + k < n$. This means that $xz$ is not a palindrome, because the numbers of zeros on the two ends don’t match.

This means that the set of palindromes doesn’t satisfy the Pumping Lemma and, thus, the set of palindromes cannot be regular.

Broken proof 4

Suppose that the set of palindromes were regular. Let $n$ be the value from the Pumping Lemma. Consider the string $w = (01)^n(10)^n$. $w$ is clearly a palindrome and $|w| \geq n$. By the Pumping Lemma, there must exist strings $x$, $y$, and $z$ satisfying the four constraints of the Pumping Lemma.

So, pick any $x$, $y$, and $z$ such that $w = xyz$, $|xy| \leq n$, and $|y| \geq 1$. Because $|xy| \leq n$, $xy$ is entirely contained in the $(01)^n$ at the start of $w$. So $y$ must be $(01)^i$ for some integer $i \geq 1$.

Now, consider $xy^2z$. $xy^2z = (01)^n(10)^n$. By the Pumping Lemma, $xy^2z$ is supposed to be in the language. But the number of $01$ and $10$ pairs don’t match (since $i \geq 1$), so $xy^2z$ can’t be a palindrome.

This means that the set of palindromes doesn’t satisfy the Pumping Lemma and, thus, the set of palindromes cannot be regular.

Since your adversary gets to pick how $w$ is divided into $x$, $y$, and $z$, $y$ might not actually be a neat set of $01$ pairs. $y$ might start with a $1$ or end with a $0$. The middle bit of this proof would need to patched up. It probably can be, since $w$ is so much longer than $n$, but it’s not going to be easy.

A better way to fix this proof would be to find a better choice for $w$.

Pumping Lemma proofs

$L = \{0^n1^m0^n \mid m, n \geq 0\}$

Assume that $L$ is regular.

Then by the Pumping Lemma for Regular Languages, there exists a pumping length $n$ for $L$ such that for any string $s \in L$ where $|s| \geq n$, $s = xyz$ subject to the following:

a. $|y| > 0$

b. $|y| \leq n$, and

c. For all $i \geq 0$, $xy^iz \in L$.

Choose $s = 0^n1^m0^n$. Clearly, $|s| \geq n$ and $s \in L$.

By condition (b) above, it follows that $x$ and $y$ are composed only of zeros.

By condition (a), it follows that $y = 0^k$ for some $k > 0$.

By condition (c), we can take $i = 0$ and $xy^iz$ should be in $L$.

$xy^2z = xz = 0^n1^m0^n$. But this string is clearly not in $L$. (why?)

This contradicts the Pumping Lemma. Therefore our assumption that $L$ is regular is incorrect, and $L$ is not a regular language.
\[ L = \{ wtw \mid w, t \in \{0,1\}^* \} \]

Assume that \( L \) is regular.

Then by the Pumping Lemma for Regular Languages, there exists a pumping length \( n \) for \( L \) such that for any string \( s \in L \) where \( |s| \geq n \), \( s = xyz \) subject to the following:

a. \( |y| > 0 \)

b. \( |xy| \leq n \), and

c. For all \( i \geq 0 \), \( xy^iz \in L \)

Choose \( s = 0^n10^n1 \). Clearly, \( |s| \geq n \) and \( s \in L \) with \( w = 0^n1 \) and \( t = 1 \).

By condition (b) above, it follows that \( x \) and \( y \) are composed only of zeros.

By conditions (a) and (b), it follows that \( y = 0^k \) for some \( k > 0 \).

By condition (c), we can take any \( i \) and \( xy^iz \) should still be in \( L \). Thus, \( xy^2z \) should be in \( L \).

\( xy^2z = xyz = 0^n(0^k110^n1) \). There is no way that this string can be divided into \( wtv \) as required to be in \( L \) (why not?), thus \( xy^2z \) is not in \( L \). This is a contradiction with condition (c) of the pumping lemma.

Therefore the assumption that \( L \) is a regular language must be incorrect and \( L \) is not a regular language.

\[ L = \{ ab^ic^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k \} \]

Assume that \( L \) is regular.

Let \( L_1 = \{ a^n \mid n \geq 100 \} \) and \( L_2 = \{ b^m \mid m \leq 100 \} \).

Then \( L = L_1 \cdot L_2 \).

We can easily prove that \( L_1 \) and \( L_2 \) are regular by constructing NFAs to recognize each of them.

Thus, we conclude that \( L = L_1 \cdot L_2 \) is regular because regular languages are closed under concatenation.

Proofs using closure properties

Prove a language regular using closure properties

\[ L = \{ a^n b^m \mid n \geq 100, m \leq 100 \} \]

Let \( L_1 = \{ a^n \mid n \geq 100 \} \) and \( L_2 = \{ b^m \mid m \leq 100 \} \).

Then \( L = L_1 \cdot L_2 \).

We can easily prove that \( L_1 \) and \( L_2 \) are regular by constructing NFAs to recognize each of them.

Thus, we conclude that \( L = L_1 \cdot L_2 \) is regular because regular languages are closed under concatenation.
Proofs using closure properties

Prove a language non-regular using closure properties

Example: \( L = a^n b^m \mid n, m \geq 0, m \neq n \)

Assume \( L \) is regular

Goal: try to construct \( \{a^n b^m\} \) which we know is not regular

Let \( L_1 = a^* b^* \), \( L_1 \) is regular

Let \( L_3 = L \)

By closure under complementation, if \( L \) is regular then \( L_3 \) is regular

By closure under intersection, if \( L_3 \) is regular, \( L_3 \cap L_1 = L_4 = a^n b^n \) is regular

Contradiction, already proved \( L_4 \) is not regular.

Thus \( L \) is not regular. QED

Proofs using closure properties

Prove a language non-regular using closure properties

\( L = \{a^n b^m c^p \mid l, m, p \geq 0 \text{ such that } l + m \neq p \} \)

Assume that \( L \) is regular.

Then its complement \( L^c \) is regular.

Intersect \( L^c \) with the regular language \( a^* b^* c^* \). The resulting language should be regular since regular languages are closed under intersection.

But the resulting language is the one in the previous example, which we have shown to be non-regular.

Therefore the assumption that \( L \) is a regular language is incorrect and \( L \) is not a regular language.

NOTE: Can also do with the Pumping Lemma with \( s = a^n b^m c^{n+m+k} \), where \( k = n! = n(n-1)(n-2)\ldots(3)(2)(1) \)